Nonisomorphic algebraic models of smooth manifolds

Kubbe, L.

2003

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VRIJE UNIVERSITEIT

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ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. T. Sminia,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der Exacte Wetenschappen
op woensdag 14 mei 2003 om 13.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

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geboren te Amsterdam

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THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

The author is member of the European Research Training Network RAAG (EC contract HPRN-CT-2001-00271)

Kubbe, Laura
Nonisomorphic algebraic models of smooth manifolds
ISBN 90-9016833-8

Printed by PrintPartners Ipskamp

Aan mijn ouders

Preface

When I started studying at the *Universiteit van Amsterdam*, I had the idea of choosing geometry as major subject. I chose to graduate under the supervision of Sebastiaan van Strien, who put me to work on elliptic geometry and the theory of Julia sets. And this way my major subject had become dynamical systems.

When van Strien left the *UvA* to hold a chair in geometry in Warwick, he referred me to Sjoerd Verduyn Lunel, who was so kind as to guide me towards my graduation. He suggested Morse theory as a subject for my Master's thesis and introduced me to Rob van der Vorst, who knew a lot about the subject. The resulting thesis was about Morse theory applied to differential equations.

By then Verduyn Lunel had moved to the *Vrije Universiteit Amsterdam* to become professor in analysis and I followed him as an *aio* (Ph.D. student). After a year, he moved on to the *Universiteit Leiden*, and although I felt very much at home at the analysis department, I thought this might be a good opportunity to find out whether it was possible to move over to the geometry department of professor Jacek Bochnak, an authority in the field of the real algebraic geometry.

Consequently I became *aio* geometry. I learned a multitude of methods and theories necessary to grasp the sophisticated constructions used in real algebraic geometry. I started visiting the RAAG, (Real Algebraic and Analytic Geometry), conferences and in course of time I gained more and more insight into the subject matter. The result is the thesis you are now holding in your hands.

It is a pleasure to acknowledge the help and support I received from Jacek Bochnak. “Jacek, thank you for taking me on as an *aio* and supervising me so splendidly. From the way you work as a mathematician, I could learn what it means to be a true researcher. Thank you for teaching me, answering all my questions, and reading everything I wrote meticulously. I can’t imagine a better way of supervising. Fantastic!”

I also like to express my gratitude to Wojciech Kucharz. His unpublished lecture notes were a great help in writing this thesis.

I am very grateful to the members of the reading committee, Johan Huisman, Wojciech Kucharz, Frédéric Mangolte, Jan van Mill, and Kamil Rusek for reading the manuscript and for their valuable suggestions for corrections and improvements.

I also thank Ieke Moerdijk and Dick Siersma for their interest in my thesis and for doing me the honor of acting as opponents.

I wish to thank my teachers of the *UvA* for their efforts and for setting such good examples. Furthermore, I thank Sjoerd Verduyn Lunel and Rob van der Vorst for their stimulating guidance during my first year at the *VUA*. Thanks are also due to Aad van der Vaart, who gave me the green light for my transfer to the geometry department.

With great joy I have worked at the mathematics department of the *Faculteit der Exacte Wetenschappen*. I would like to express my appreciation to the staff for a fine and relaxed atmosphere in which I felt very much at home. I like to thank Maryke Titawano for always being available for support. Special thanks go to Ruud Wiggers for providing me with a notebook.

I would like to thank my fellow *aio's* for being such good company. I wish to give special thanks to the chairwoman of the tea club, Martine Reurings, as well as to the her preceding tea masters, who could also pour a nice cup of tea.

Finally, I thank Martine Reurings and Hendrik Blauwendraat for doing me the honor of acting as *paranimfen* during the presentation of this thesis.

Contents

Introduction

In real algebraic geometry the notion of nonsingular algebraic variety corresponds to the notion of smooth manifold in topology. It follows from the definition of nonsingularity that a nonsingular algebraic set is a smooth manifold. If M is a smooth manifold and X a nonsingular affine real algebraic variety diffeomorphic with M, then we call X an *algebraic model of* M.

The question whether each compact smooth manifold M admits an algebraic model, has been answered affirmatively in 1973 by Tognoli [38]. His theorem is a generalization of a result of Nash, who proved in 1952 that every compact connected smooth manifold is diffeomorphic to a connected component of a nonsingular real algebraic set [32].

Theorem 1 (Nash's theorem) *Let $M \subset \mathbb{R}^m$ be a compact connected smooth submanifold. Then M is diffeomorphic to a nonsingular connected component of an algebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$ for some n.*

In order to prove that M is in fact diffeomorphic to a connected algebraic set and not just to a component, one needs to apply the theory of *cobordism.* The idea to use cobordism to investigate the question of the existence of algebraic models first appeared in an article by Wallace published in 1957 [39]. If M_0 and M_1 are two compact smooth m-dimensional manifolds, then they are called *cobordant* if their disjoint union $M_0 \sqcup M_1$ is the boundary of a compact smooth $(m+1)$-dimensional manifold with boundary. What is needed for Tognoli's result, is that every compact smooth manifold is cobordant to a nonsingular compact real algebraic set, cf. [26] or [37]. Furthermore, one needs a more precise version of Nash's theorem [3, Thm. 14.1.7].

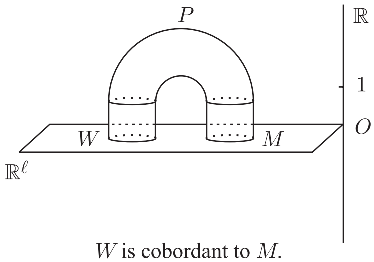
Theorem 2 (The generalized Nash theorem) *Let $M \subset \mathbb{R}^m$ be a connected compact smooth submanifold. Let $A \subset M$ be a nonsingular algebraic subset, and assume that some open neighborhood U of A in M is an open subset of a nonsingular algebraic subset of $\mathbb{R}^m$. Then there exists a nonsingular connected component X of an algebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$ (for some n) and a smooth diffeomorphism $\varphi : M \to X$, such that $\varphi(a) = (a, 0)$ for every $a \in A$.*

We outline how all these ingredients lead to Tognoli's result, cf. [3, Thm. 14.1.10].

Theorem 3 (Tognoli's theorem) *Let M be a compact smooth connected submanifold of $\mathbb{R}^\ell$. Then M is diffeomorphic to a nonsingular algebraic subset of $\mathbb{R}^p$, for some $p \geq \ell$.*

Sketch of the proof. As mentioned earlier, every compact smooth manifold is cobordant to a nonsingular compact real algebraic set. So let W be a nonsingular compact algebraic set cobordant to M, that is, there exists a compact smooth $(\ell+1)$-dimensional manifold P with boundary the disjoint union $W \sqcup M$. We may assume that the manifold P is contained in $\mathbb{R}^\ell \times [0, \infty)$ such that

$$\partial P = P \cap \left(\mathbb{R}^{\ell} \times \{0\}\right) \text{ and } P \cap \left(\mathbb{R}^{\ell} \times [0,1)\right) = \partial P \times [0,1).$$

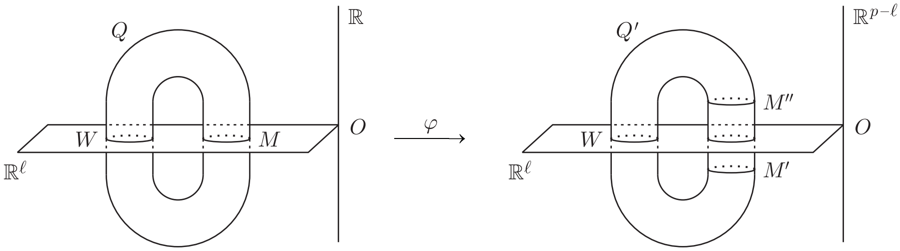


W is cobordant to M.

Define the smooth manifold

$$Q = \left\{ (v,r) \in \mathbb{R}^{\ell} \times \mathbb{R} \;\middle|\; (v,r) \in P \quad \text{or} \quad (v,-r) \in P \right\}.$$

So Q is the "double" of P. Then by the generalized Nash theorem, there exists a nonsingular connected component Q' of some algebraic set $Z \subset \mathbb{R}^p$, with $p \geq \ell + 1$, and a diffeomorphism $\varphi : Q \to Q'$ such that $\varphi(W) = W$. Set $M' = \varphi(M)$.



Q is diffeomorphic to Q' and M' is diffeotopic to M''.

Let $g : Q' \to \mathbb{R}$ be a smooth function such that $g^{-1}(0) = W \cup M'$

and 0 is a regular value of g (take the composition of $\varphi^{-1} : Q' \to Q$ with the mapping $Q \to \mathbb{R}$, $(v, r) \mapsto r$). We can choose a regular function $h : Z \to \mathbb{R}$ such that $h|_W = 0$, and $h^{-1}(0) \subset Q'$. This function h can be chosen in such a way that $h^{-1}(0)$ and $g^{-1}(0) = W \cup M'$ are diffeotopic, with a diffeotopy of Q' keeping W fixed. It follows that $h^{-1}(0) = W \cup M''$ with M'' diffeotopic to M'. Since W and $W \cup M''$ are nonsingular algebraic sets with the same dimension, it follows that M'' is a nonsingular algebraic set and clearly M'' is diffeomorphic to M.

□

The algebraic model of M obtained above is not unique. In 1991, Bochnak and Kucharz showed that every smooth manifold M even admits uncountably many mutually nonisomorphic algebraic models [6]. To distinguish between nonisomorphic algebraic models of a smooth manifold, we use the fact that there exist uncountably many nonisomorphic elliptic curves defined over $\mathbb{R}$ (nonsingular complex projective cubics defined over $\mathbb{R}$ equipped with an appropriate group operation). More precisely, given a nonsingular projective real algebraic variety X, there exists a complex abelian variety $\mathrm{Alb}(X)$ (called the *Albanese variety* of X) such that the existence of a nonconstant regular mapping $f : X \to C$ from X into the real part C of a complex elliptic curve E defined over $\mathbb{R}$, implies the existence of a surjective homomorphism $\tilde{f} : \mathrm{Alb}(X) \to E$. The Albanese variety is a birational invariant of X. This gives us a tool to distinguish between nonisomorphic (or even birationally nonequivalent) varieties. Indeed, let $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ be an uncountable family of complex elliptic curves in $\mathbb{C}P^2$ defined over $\mathbb{R}$ such that E_α is not isogenous to E_β for $\alpha \neq \beta$ and let C_α be the real part of E_α. If $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of affine nonsingular real algebraic varieties, and for each $\alpha \in \mathcal{A}$ there exists a nonconstant regular mapping $f_\alpha : X_\alpha \to C_\alpha$, then $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ contains an uncountable family $\{X_\alpha\}_{\alpha\in\mathcal{B}}$ of mutually nonisomorphic varieties (cf. Theorem 4.1.2).

Let X be a compact nonsingular n-dimensional real affine algebraic variety. We denote by $H_{n-1}^{\mathrm{alg}}(X,\mathbb{Z}/2)$ the subgroup of $H_{n-1}(X,\mathbb{Z}/2)$ consisting of all homology classes represented by Zariski closed $(n-1)$-dimensional subvarieties of X and we denote by $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ the subgroup defined by

$$H^1_{\mathrm{alg}}(X,\mathbb{Z}/2) = D_X^{-1}(H_{n-1}^{\mathrm{alg}}(X,\mathbb{Z}/2))$$

where D_X is the Poincaré duality isomorphism (see Section 2.3). The group $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ of cohomology classes represented by algebraic cycles of codimension 1 on X is a very important invariant of X. In [5] Bochnak and Kucharz study this group $H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)$ as X runs through the class of algebraic models of M and obtain the following result.

Theorem 4 *Let M be a compact connected smooth manifold of dimension ≥ 3. Given a subgroup G of $H^1(M,\mathbb{Z}/2)$ containing the first Stiefel-Whitney class of M, there exists an algebraic model X of M and a diffeomorphism $\varphi : X \to M$ with $\varphi^*(G) = H_1^{\mathrm{alg}}(X,\mathbb{Z}/2)$.*

A natural question which arises is: Does there exist an uncountable family of mutually nonisomorphic algebraic models of M with this property? In Chapter 4 of this thesis we show that this is indeed the case and we can formulate this as follows.

Theorem 4.3.2 *Let M be a compact connected smooth manifold of dimension $m \geq 3$ and let G be a subgroup of $H^1(M,\mathbb{Z}/2)$ containing* *the first Stiefel-Whitney class of M. Then there exists an uncountable family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ of algebraic models of M and a family of smooth diffeomorphism $\{h_\alpha : X_\alpha \to M\}_{\alpha\in\mathcal{A}}$ such that*

(i) X_α and X_β are not birationally equivalent for $\alpha \neq \beta$,

(ii) $h_\alpha^*(G) = H^1_{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2)$ for each $\alpha \in A$.

The proof requires some rather sophisticated constructions. As in Tognoli's theorem we first choose a suitable manifold P containing M. Then we construct an algebraic set Y of M such that the triple $Y \subset M \subset P$ satisfies a number of conditions which are needed to find a nonsingular variety diffeomorphic to M. The theorem which makes it possible to find this algebraic model (Theorem 3.2.8) is an important algebraic approximation theorem of Bochnak and Kucharz, and it is in the proof of this theorem that bordism theory appears as an essential tool (as cobordism theory does in the proof of Tognoli's theorem). We introduce an uncountable family $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ of mutually nonisogenous complex elliptic curves defined over $\mathbb{R}$ with their real parts $\{C_\alpha\}_{\alpha\in\mathcal{A}}$. Then by the algebraic approximation theorem, the conditions mentioned above imply the existence of uncountably many algebraic models X_α of M together with nonconstant regular mappings $f_\alpha : X_\alpha \to C_\alpha$. Having these regular mappings, we can distinguish the nonisomorphic algebraic models X_α by applying the method involving the Albanese varieties $\mathrm{Alb}(X_\alpha)$ described earlier. Finally, we use the properties of the groups of the cohomology classes represented by algebraic cycles of codimension 1 on M, X_α, and Y in such a way that part (ii) of the theorem follows.

For smooth compact connected surfaces a similar result can be found. For instance, if M is a compact connected smooth orientable surface of genus g, and k is an integer such that $0 \leq k \leq 2g$, then there exists an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraic models of M such that the dimension of the $\mathbb{Z}/2$ vector space $H^1_{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2)$ is equal to k.

An interesting question is: Given two nonsingular affine real algebraic varieties X and Y, what can we say about the size of the set of all smooth mappings $X \to Y$ which are approximable by regular mappings? In section 5.1 we apply Theorem 4.3.2 and the results of Section 4.4 to investigate the size of the set of regular mappings from X into the unit circle S^1, where X runs through an uncountable family of nonisomorphic algebraic models X of a given smooth manifold (cf. Theorem 5.1.3). As another application of the results of Chapter 4, we can prove that for each $n \geq 2$, there exists an uncountable family of mutually nonisomorphic Noetherian factorial rings $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of Krull dimension n such that each A_α has the property that $A_\alpha/\mathfrak{m} \simeq \mathbb{R}$ for each maximal ideal $\mathfrak{m}$ of A_α.

The theory that enables us to obtain these results is given in the first three chapters of this thesis. Chapter 1 presents some basic notions and results in real algebraic geometry. In the last section Albanese varieties are introduced and a number of properties necessary for our purposes are discussed. Chapter 2 deals with the representation of cohomology classes by real algebraic subvarieties and discusses the connection with algebraic bordism theory. Chapter 3 is concerned with algebraic approximations of smooth submanifolds of affine nonsingular real algebraic varieties and contains the proof of an algebraic approximation theorem which is of great importance for our findings. The main results are established in Chapter 4 and two applications are given in Chapter 5. In Appendix A we give a few results concerning Thom classes, which are needed for the proofs of Theorem 2.1.6 and Corollary 2.1.7 in Section A.3. A result which we use in the proof of Theorem 2.4.3 is proved in Appendix B.

1 Preliminaries

In this chapter we present some basic notions and results in algebraic geometry that we will be using later on.

1.1 Algebraic sets

A subset $V \subset \mathbb{R}^n$ is called an *algebraic set* if there exists a family $\mathcal{F}$ of polynomials in $\mathbb{R}[T_1, \ldots, T_n]$ such that

$$V = \{ \ x \in \mathbb{R}^n \ \mid P(x) = 0 \text{ for all } P \in \mathcal{F} \ \}.$$

By Hilbert's basis theorem [40, Vol I, p. 201], V is the set of common zeros of finitely many polynomials in $\mathbb{R}[T_1, \ldots, T_n]$. The algebraic subsets of $\mathbb{R}^n$ form the closed sets of a topology on $\mathbb{R}^n$, called the *Zariski topology*. The space $\mathbb{R}^n$ can also be endowed with the Euclidean topology, that is, the topology induced by the usual metric on $\mathbb{R}^n$. Each algebraic subset is therefore endowed with two topologies: the Zariski topology and the Euclidean topology.

If $V \subset \mathbb{R}^n$ is an algebraic set then we denote by

$$I(V) = \{f \in \mathbb{R}[T_1, \ldots, T_n] \ \mid f(x) = 0 \text{ for all } x \in V\}$$

the ideal of $\mathbb{R}[T_1, \ldots, T_n]$ of polynomials vanishing on V. A nonempty algebraic set V is called *irreducible* if it cannot be expressed as the union of two proper algebraic subsets. Note that an algebraic set V is irreducible if and only if $I(V)$ is a prime ideal [18, p. 4]. Each nonempty algebraic set V can be expressed as a finite union $V = V_1 \cup \ldots \cup V_k$ of irreducible algebraic sets. If we require that $V_i \not\subset V_j$ for $i \neq j$, then the V_i are uniquely determined [18, p. 5]. They are called the *irreducible components* of V.

Let V be an algebraic subset of $\mathbb{R}^n$. A *polynomial function* $f : V \to \mathbb{R}$ is the restriction to V of a polynomial $F \in \mathbb{R}[T_1, \ldots, T_n]$. We denote the ring of polynomial functions by $\mathcal{P}(V)$.

Definition 1.1.1 *Let $V \subset \mathbb{R}^n$ be an algebraic set, and let U be a Zariski open subset of V. A function $f : U \to \mathbb{R}$ is called a* regular function *if there exist polynomials P and Q in $\mathcal{P}(V)$ such that*

$$f(x) = \frac{P(x)}{Q(x)} \text{ and } Q(x) \neq 0 \text{ for all } x \in U.$$

The ring of regular functions on U is denoted by $\mathcal{R}(U)$. Assume that $V \subset \mathbb{R}^n$ is an irreducible algebraic set. The quotient field of the integral ring $\mathbb{R}[T_1, \ldots, T_n] / I(V)$ is called the *field of rational functions* on V and is denoted by $\mathcal{K}(V)$. The polynomial ring $\mathcal{P}(V)$ is isomorphic to the ring $\mathbb{R}[T_1, \ldots, T_n] / I(V)$, since $I(V)$ is equal to the kernel of the surjective ring homomorphism

$$\varphi : \mathbb{R}[T_1, \ldots, T_n] \to \mathcal{P}(V), \;\; \varphi(F) = f,$$

where $f|_V = F$. A regular function is thus a rational function without poles.

Now suppose that $U = \bigcup_{i=1}^{p} U_i$ with $U_i \subset U$ Zariski open. If $f : U \to \mathbb{R}$ is a function on U such that $f|_{U_i} \in \mathcal{R}(U_i)$ for $i = 1, \ldots, p$, then it follows that $f \in \mathcal{R}(U)$ [3, Prop. 3.2.3]. This implies that the assignment $\mathcal{R}_V : U \to \mathcal{R}(U)$ is a sheaf of rings on V. The pair $(V, \mathcal{R}_V)$ is then a *locally ringed space* [18, p. 72].

Let $V \subset \mathbb{R}^n$ be algebraic set and let $I(V)$ be the ideal generated by the polynomials $f_1, \ldots, f_k$ in $\mathbb{R}[T_1, \ldots, T_n]$. The *Zariski tangent space* of V at a point $z \in V$ is defined by

$$T_z^{\mathrm{Zar}}(V) = \bigcap_{j=1}^{k} \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} \frac{\partial f_j(z)}{\partial x_i} x_i = 0 \right\}.$$

Assume that the set V is irreducible. A point $z \in V$ is called a *nonsingular point* if $\dim T_z^{\mathrm{Zar}}(V) = \dim(V)$, where the dimension of V is defined to be the transcendence degree over $\mathbb{R}$ of the field of rational functions $\mathcal{K}(V)$, cf. [40]. It follows that z is nonsingular if and only if the matrix

$$\left(\frac{\partial f_j}{\partial x_i}(z) \right)_{i,j}$$

has rank equal to $n - \dim(V)$.

Proposition 1.1.2 *Let $V \subset \mathbb{R}^n$ be an irreducible algebraic set of dimension d and let $z \in V$ be a nonsingular point of V. Then there exist $n-d$ polynomials $f_1, \ldots, f_{n-d}$ in $I(V)$ and a Zariski open neighborhood U of z in $\mathbb{R}^n$, such that*

(*i*) $V \cap U = W \cap U$,
where $W = \{x \in \mathbb{R}^n \mid f_1(x) = \ldots = f_{n-d}(x) = 0\}$,

(*ii*) *for each* $x \in U$, $\operatorname{rank} \left(\frac{\partial f_j}{\partial x_i}(x) \right)_{i,j} = n - d$.

A proof may be found in [3, p. 67]. From Proposition 1.1.2 and the inverse function theorem it follows that a nonsingular algebraic set is a smooth manifold of dimension d.

Let V be an algebraic set such that $V = V_1 \cup \ldots \cup V_k$ is the union of irreducible components V_j. Then a point $x \in V$ is called *non-singular* if it belongs to a single irreducible component V_j for some j, $\dim V = \dim V_j$, and x is a nonsingular point of V_j. An algebraic set is said to be *nonsingular* if all its points are nonsingular. We denote the set of all nonsingular points of an algebraic subset V of $\mathbb{R}^n$ by $\mathrm{Reg}\,(V)$. The set

$$\mathrm{Sing}\,(V) = V \setminus \mathrm{Reg}\,(V)$$

of all singular points of V is a proper algebraic subset of V (possibly empty) with $\dim\,(\mathrm{Sing}\,(V)) < \dim V$ [3, p. 69].

For our purposes it is useful to consider also algebraic sets in $\mathbb{C}^n$ and in $\mathbb{C}P^n$. We call a subset $V \subset \mathbb{C}^n$ a *complex algebraic set* if V is the common zero set of a collection of polynomials in $\mathbb{C}\,[T_1, \ldots, T_n]$. A subset $V \subset \mathbb{C}^n$ is said to be an *algebraic set defined over* $\mathbb{R}$ if there exists a family $\mathcal{F}$ of polynomials in $\mathbb{R}\,[T_1, \ldots, T_n]$ such that

$$V = \{\ x \in \mathbb{C}^n \ \mid P(x) = 0 \text{ for all } P \in \mathcal{F}\ \}.$$

A subset of the complex projective space $\mathbb{C}P^n$ which is the common zero set of a collection of homogeneous polynomials in $\mathbb{C}\,[T_0, \ldots, T_n]$ is called a *projective complex algebraic subset*. A subset $V \subset \mathbb{C}P^n$ is called a *projective algebraic subset defined over* $\mathbb{R}$ if there exists a family $\mathcal{F}$ of homogeneous polynomials in $\mathbb{R}\,[T_0, \ldots, T_n]$ such that

$$V = \{\ (x_0 : \ldots : x_n) \in \mathbb{C}P^n \ \mid P\,(x_0 : \ldots : x_n) = 0 \text{ for all } P \in \mathcal{F}\ \}.$$

The Zariski topology on $\mathbb{C}^n$ or $\mathbb{C}P^n$ is defined by taking algebraic sets as closed sets. We denote the space $\mathbb{C}^n$ endowed with the Zariski topology of algebraic sets defined over $\mathbb{R}$ (resp. complex algebraic sets) by $\mathbb{A}^n_{\mathbb{R}}(\mathbb{C})$ (resp. $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C})$). Similarly, we denote the space $\mathbb{C}P^n$ endowed with the Zariski topology of algebraic sets defined over $\mathbb{R}$ by $\mathbb{P}^n_{\mathbb{R}}(\mathbb{C})$ and $\mathbb{C}P^n$ endowed with the Zariski topology of complex algebraic sets by $\mathbb{P}^n_{\mathbb{C}}(\mathbb{C})$.

Definition 1.1.3 *Let U be a Zariski open subset of a Zariski closed subset of $\mathbb{A}^n_{\mathbb{R}}(\mathbb{C})$. A complex function $f : U \to \mathbb{C}$ is called a* regular function defined over $\mathbb{R}$, *if there exist polynomials $P, Q \in \mathbb{R}[T_1, \ldots, T_n]$ such that*

$$f(x) = \frac{P(x)}{Q(x)} \text{ and } Q(x) \neq 0 \text{ for all } x \in U.$$

Let U be a Zariski open subset of $\mathbb{P}^n_{\mathbb{R}}(\mathbb{C})$. A function $f : U \to \mathbb{C}$ is called a regular function defined over $\mathbb{R}$ *if each point $x \in U$ has a neighborhood U' in U such that there exist homogeneous polynomials $P, Q \in \mathbb{R}[T_0, \ldots, T_n]$ having the same degree, such that*

$$f(x_0, \ldots, x_n) = \frac{P(x_0, \ldots, x_n)}{Q(x_0, \ldots, x_n)} \text{ and}$$

$$Q(x_0, \ldots, x_n) \neq 0 \quad \text{for all } x = (x_0 : \ldots : x_n) \in U'.$$

Replacing $\mathbb{R}$ by $\mathbb{C}$ in the definitions above, one obtains the analogous objects defined over the field $\mathbb{C}$.

1.2 Regular and rational mappings

Definition 1.2.1 *Let $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^k$ be two algebraic sets, and let U and U' be Zariski open subsets of V and V', respectively.*

Then the mapping $\varphi : U \to U'$ is called a regular mapping *if each real valued component φ_i of $\varphi = (\varphi_1, \ldots, \varphi_k)$ is a regular function. The mapping $\varphi : U \to U'$ is called a* biregular isomorphism, *if φ is bijective and φ and φ^{-1} are both regular.*

If $V \to V'$ is a biregular isomorphism and V is irreducible, then clearly V' is also irreducible. Furthermore, the image of a nonsingular point of V is nonsingular in V' [3, p. 66].

An important equivalence relation for the classification of real algebraic sets is the *birational equivalence.*

Definition 1.2.2 *Let V be an irreducible real algebraic set and V' any real algebraic set. A* rational mapping *$\varphi : V \to V'$ is an equivalence class of pairs (U, f), where U is a nonempty Zariski open subset of V and $f : U \to V'$ is a regular mapping. Two pairs (U, f) and (W, g) are said to be* equivalent *if $f = g$ on $U \cap W$.*

A rational mapping $\varphi : V \to V'$ may not be defined at some points of V. The union U of all Zariski open subsets U_i of V such that $f_i : U_i \to V'$ belongs to the equivalence class of φ, is called the *domain* of φ and is denoted by $\operatorname{dom}(\varphi)$. The mapping $f : U \to V'$ defined by $f|_{U_i} = f_i$ is then a regular mapping belonging to the equivalence class of φ.

Since rational mappings fail to behave like ordinary mappings, the composition of two rational mappings is not always defined. If $\varphi : V \to V'$ and $\psi : V' \to V''$ are two rational mappings, then the composition $\varphi \circ \psi$ is well-defined if $f(V)$ is Zariski dense in V' for some pair (U, f) defining φ.

Definition 1.2.3 *A rational mapping $\varphi : V \to V'$ is called* birational *if there exists a rational mapping $\psi : V' \to V$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are both defined and equal to* id_V *and* $\mathrm{id}_{V'}$*, respectively, as rational mappings. We say that V and V' are* birationally equivalent *if there exists a birational mapping from V to V'.*

If V and V' are birationally equivalent, then there are Zariski open subsets $U \subset V$ and $U' \subset V'$ such that U and U' are isomorphic. Indeed, let $\varphi : V \to V'$ be a rational mapping and $\psi : V' \to V$ its inverse. Let φ be represented by (U, f) and ψ by (U', g). Then $\psi \circ \varphi$ is represented by $\left(f^{-1}(U'), g \circ f\right)$. Since $\psi \circ \varphi = \mathrm{id}_V$ as a rational mapping, $g \circ f$ is the identity on $f^{-1}(U')$. Similarly, $f \circ g$ is the identity on $g^{-1}(U)$. It follows that the Zariski open sets $f^{-1}\left(g^{-1}(U)\right) \subset V$ and $g^{-1}\left(f^{-1}(U')\right) \subset V'$ are biregularly isomorphic.

1.3 Algebraic varieties

The main objects of interest in real algebraic geometry are affine real algebraic varieties and their morphisms. Let X and Y be topological spaces and let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be sheaves of real valued functions on X and Y, respectively. In general, a morphism from the ringed space $(X, \mathcal{O}_X)$ into the ringed space $(Y, \mathcal{O}_Y)$ is a continuous mapping $\varphi : X \to Y$ such that for every open subset V of Y and for every function g in $\mathcal{O}_Y(V)$, the composite function $g \circ (\varphi|_{g^{-1}(V)})$ belongs to $\mathcal{O}_X(\varphi^{-1}(V))$.

A morphism is called an *isomorphism* if it has an inverse mapping which is a morphism.

Definition 1.3.1 *An* affine real algebraic variety *is a topological space X equipped with a sheaf $\mathcal{O}_X$ of real valued functions, isomorphic (as a ringed space) to an algebraic set $V \subset \mathbb{R}^n$ with its Zariski topology and equipped with its sheaf of regular functions $\mathcal{R}_V$.*

A real algebraic variety *is a topological space X equipped with a sheaf $\mathcal{O}_X$ of real valued functions such that there exists a finite open cover $\{U_i\}_{i \in I}$ of X, with each $(U_i, \mathcal{O}_X|_{U_i})$ being an affine real algebraic variety.*

The sections of $\mathcal{O}_X$ are called *regular functions* on X and the topology on X is called the *Zariski topology.* If X and Y are two real algebraic varieties with sheaves of regular functions $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively, then a *regular mapping* $\varphi : X \to Y$ is a morphism of ringed spaces. We denote the set of all regular mappings from X into Y by $\mathcal{R}(X, Y)$. A regular mapping $\varphi : X \to Y$ is called a *biregular isomorphism* if φ has a regular inverse.

Example 1.3.2 *A real algebraic variety which will be very useful is the Grassmannian. We shall denote this variety by $G_{n,k}(\mathbb{R})$. As a set, the Grassmannian $G_{n,k}(\mathbb{R})$ is defined to be the set of all vector subspaces of dimension k of $\mathbb{R}^n$. In particular, we have $G_{n+1,1}(\mathbb{R}) = \mathbb{R}P^n$.*

Denote by $M(n, m; \mathbb{R})$ the set of $n \times m$ matrices with coefficients in $\mathbb{R}$. Clearly, $M(n, m; \mathbb{R})$ is an affine real algebraic variety isomorphic to $\mathbb{R}^{nm}$. The Grassmannian $G_{n,k}(\mathbb{R})$ can be equipped with the structure of a nonsingular real algebraic variety in such a way that $G_{n,k}(\mathbb{R})$ is *biregularly isomorphic to the algebraic set*

$$H_{n,k} = \left\{ A \in M(n, n; \mathbb{R}) \mid {}^{t}A = A^2 = A,\ \operatorname{trace}(A) = k \right\}.$$

The biregular isomorphism is obtained by sending the vector space $V \in G_{n,k}(\mathbb{R})$ *to the matrix of the orthogonal projection onto* V*. Thus the Grassmannian* $G_{n,k}(\mathbb{R})$ *is a nonsingular affine real algebraic variety [3, p. 71-72].*

By a *complex algebraic variety defined over* $\mathbb{R}$ we understand a quasiprojective subvariety of $\mathbb{P}^n_{\mathbb{R}}(\mathbb{C})$ defined over $\mathbb{R}$, equipped with the sheaf of regular functions defined over $\mathbb{R}$.

Unless explicitly stated, all topological notions related to algebraic varieties will refer to the Euclidean topology.

Definition 1.3.3 *A nonsingular affine real algebraic variety* X *diffeomorphic to a smooth manifold* M *is called an* algebraic model of M*.*

By a well-known theorem of Nash and Tognoli [32][38], every compact smooth manifold admits an algebraic model.

Let X be an affine real algebraic variety contained in $\mathbb{R}^n$ for some n, and consider $\mathbb{R}^n$ as a subset of the projective space $\mathbb{R}P^n \subset \mathbb{C}P^n$. By the *Zariski closure* of X in $\mathbb{C}P^n$ we mean the smallest algebraic subset of $\mathbb{C}P^n$ containing X. We define the *nonsingular projective complexification* of X, cf. [9, p. 264].

Definition 1.3.4 *If* X *is a nonsingular projective real algebraic variety, then a* nonsingular projective complexification *of* X *is a pair* (V, j)*, where* V *is a nonsingular projective complex algebraic subset of* $\mathbb{C}P^n$ *defined over* $\mathbb{R}$ *and* $j : X \to V$ *is an injective mapping such that* (*i*) $j(X) = V \cap \mathbb{R}P^n$,

(*ii*) *the Zariski closure of* $j(X)$ *in* $\mathbb{C}P^n$ *is equal to* V, *and*

(*iii*) *the mapping* j, *viewed as a mapping from* X *onto* $V \cap \mathbb{R}P^n$, *is an isomorphism of real algebraic varieties.*

By Hironaka's desingularization theorem [19], X always has a nonsingular projective complexification. We will denote a nonsingular projective complexification of X by $X'_{\mathbb{C}}$. The variety $X'_{\mathbb{C}}$ is unique up to birational isomorphism defined over $\mathbb{R}$.

1.4 Vector bundles

Let M be a smooth manifold and let $V \subset M$ be a smooth submanifold of M. If $\xi = (E, \pi, V)$ is a real vector bundle over V, then we denote by O_E the image of the zero section $V \to E$ which assigns to each $x \in V$ the zero element 0_{E_x} of the fibre $E_x = \pi^{-1}(x)$.

Definition 1.4.1 *Let* M *be a smooth manifold and let* $V \subset M$ *be a submanifold. An* open tubular neighborhood of V in M *is a smooth vector bundle* $\xi = (E, \pi, V)$ *over* V, *together with a smooth embedding* $f : E \to M$ *such that*

(*i*) $f|_V = id_V$ *where* V *is identified with the zero section ;*

(*ii*) $f(E)$ *is an open neighborhood of* V *in* M.

Tubular neighborhoods are not unique, but any two tubular neighborhoods of a submanifold are isotopic [20, p. 112].

Definition 1.4.2 *Let M be a smooth submanifold of $\mathbb{R}^m$ and let V be a smooth submanifold of M. The* normal bundle $\nu_{V,M}$ *of V in M is the vector bundle with total space*

$$E(\nu_{V,M}) = \{(p,x) \in V \times \mathbb{R}^m \mid x \in T_pM \quad \textit{and} \quad x \perp T_pV\},$$

where T_pM and T_pV are the tangent spaces at p to M and V, respectively, and projection mapping

$$\pi : E(\nu_{V,M}) \to V, \quad (p,x) \mapsto p.$$

If the bundle $\nu_{V,M}$ is trivial, then we call V a submanifold of M *embedded with a trivial normal vector bundle.* Denote by $\tau_M|_V$ the restriction of the tangent bundle τ_M to V. Then τ_V is a subbundle of $\tau_M|_V$ and its orthogonal complement is the normal bundle $\nu_{V,M}$ of V in M:

$$\tau_V \oplus \nu_{V,M} = \tau_M|_V \ .$$

If $V \subset M$ is a neat submanifold, that is, if $\partial V = V \cap \partial M$ and V is covered by charts (φ, U) of M such that $V \cap U = \varphi^{-1}(\mathbb{R}^m)$ where $m = \dim V$, then V always has a tubular neighborhood in M (cf. [20, p. 30, 114]). In particular, one can prove the following theorem.

Theorem 1.4.3 *(Tubular neighborhood theorem) Let M be a smooth submanifold of $\mathbb{R}^m$, possibly with boundary, and let V be a smooth submanifold of M. Let $\nu_{V,M}$ be the normal vector bundle of V in M and identify V with $V \times \{0\}$ in the total space $E(\nu_{V,M})$. Assume that $\partial M \cap V = \emptyset$. Then there exist a neighborhood T of V in M, a neighborhood T' of V in $E(\nu_{V,M})$, and two mappings $\theta : T \to T'$ and $\rho : T \to V$ such that* (*i*) *T and T' are smooth submanifolds with boundary, of M and $E(\nu_{V,M})$, respectively,*

(*ii*) *The mapping ρ is a smooth retraction onto V,*

(*iii*) *The mapping θ is a diffeomorphism, $\theta(p) = p$ for all p in V, and $\rho = \pi \circ \theta$.*

The pair (T, ρ) is called a *closed normal tubular neighborhood of V in M*. Let $\pi' : E(\nu_{V,M}) \to \mathbb{R}^m$ be the mapping defined by

$$\pi'(x, v) = v$$

and let $\theta' = \pi' \circ \theta : T \to \mathbb{R}^m$. The mapping θ' is called the *orthogonalization mapping*. Observe that $\theta'(t) = 0$ if and only if $t \in V$, and $\theta'(t)$ is a vector tangent to M at $\rho(t)$ and perpendicular to $T_{\rho(t)}V$. If $M = \mathbb{R}^m$, then one can take $\theta^{-1}(x, v) = x + v$ for $(x, v) \in T' \subset V \times \mathbb{R}^m$ and $\theta'(t) = \rho(t) - t$ for $t \in T$. Theorem 1.4.3 can be found in [12, p. 92] or in [28, p. 115], although it is slightly modified here to serve our purposes.

A well known result is the *regular value theorem*: if $f : M \to N$ is a smooth mapping between smooth manifolds, and q is a regular value of f, then $f^{-1}(q)$ is a smooth submanifold of M [12, p. 84], [19, p. 22]. Also the following is well known.

Theorem 1.4.4 *Let Y be a compact smooth submanifold of $\mathbb{R}^n$ of codimension 1. Then there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that 0 is a regular value of f and $f^{-1}(0) = Y$.*

Remark 1.4.5 *A similar result is that if M is an m-dimensional smooth manifold and W is an m-dimensional closed submanifold of M with boundary Y (so that Y has codimension 1), then there exists a smooth function $f : M \to \mathbb{R}$ such that 0 is a regular value of f and $f^{-1}(0) = Y$.*

Now consider the case where Y has codimension $k \geq 1$ (see also [5, Prop. 4.2]).

Lemma 1.4.6 *Let V be a closed submanifold of a smooth manifold M, with $\partial V = \emptyset$ and $\operatorname{codim} V = k$. If the normal vector bundle of V in M is trivial, then there exists a neighborhood U of V in M and a smooth function $\psi : U \to \mathbb{R}^k$ such that 0 is a regular value of ψ and $\psi^{-1}(0) = V$.*

Proof. Let U be a tubular neighborhood of V in M. Since V has a trivial normal vector bundle in M, U is diffeomorphic to $V \times \mathbb{R}^k$. Let $f : U \to V \times \mathbb{R}^k$ be a diffeomorphism satisfying $f(v) = (v, 0)$ for $v \in V$, and let π be the projection $V \times \mathbb{R}^k \to \mathbb{R}^k$. Then the mapping

$$\psi : U \to \mathbb{R}^k, \quad \psi(u) = (\pi \circ f)(u)$$

has 0 as a regular value, and $V = \psi^{-1}(0)$. □

Proposition 1.4.7 *Let Y be a closed submanifold of a smooth manifold M, with $\operatorname{codim} Y = k$. Then the following conditions are equivalent:*

- *(i) There exists a smooth mapping $\varphi : M \to \mathbb{R}^k$ such that $0 \in \mathbb{R}^k$ is a regular value of φ and $\varphi^{-1}(0) = Y$.*
- *(ii) Y bounds a closed submanifold V of M embedded with a trivial normal vector bundle.*

Proof. $(i) \Rightarrow (ii)$ Let $\varphi = (\varphi_1, \ldots \varphi_k)$ and let $\gamma : M \to \mathbb{R}$ be a smooth function such that $\gamma^{-1}(0) = Y$. For almost all $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, the points $0 \in \mathbb{R}^k$ and $0 \in \mathbb{R}^{k-1}$ are regular values of the mappings

$$\psi = (\varphi_1 + c_1\gamma, \ldots \varphi_k + c_k\gamma) \text{ and } \tilde{\psi} = (\varphi_1 + c_1\gamma, \ldots \varphi_{k-1} + c_{k-1}\gamma),$$

respectively (Sard's theorem). If c is chosen sufficiently small, then it follows from Thom's isotopy theorem (Theorem 3.2.4, [37, p. 51]) that $\psi^{-1}(0) = \varphi^{-1}(0) = Y$. Now let V be the set

$$V = \{ x \in M \mid \psi_1(x) = \ldots = \psi_{k-1}(x) = 0 \text{ and } \psi_k(x) \geq 0\} .$$

Then V is a closed submanifold of M with boundary $\partial V = Y$.

Since $0 \in \mathbb{R}^{k-1}$ is a regular value of the mapping $\tilde{\psi} : M \to \mathbb{R}^{k-1}$, the inverse image $\tilde{\psi}^{-1}(0) = \tilde{V}$ is a smooth submanifold with trivial normal bundle in M. Furthermore, V is a smooth submanifold (with boundary) of $\tilde{V}$ and of the same dimension as $\tilde{V}$. This implies that the normal bundle $\nu_{V,M}$ of V in M is trivial.

$(ii) \Rightarrow (i)$ If $\operatorname{codim} Y = k = 1$, then the implication follows from Remark 1.4.5.

Assume $k \geq 2$. Let V be as in (ii). We can choose an open tubular neighborhood U of V in M and a smooth submanifold V' of U, such that $\operatorname{codim} V = \operatorname{codim} V' = k - 1$, $V \subset V'$, $\partial V' = \emptyset$, and V' has a trivial normal vector bundle in U. Then it follows from Lemma 1.4.6 that there exists a smooth mapping $\psi = (\psi_1, \ldots, \psi_{k-1}) : U \to \mathbb{R}^{k-1}$ such that $0 \in \mathbb{R}^{k-1}$ is a regular value of ψ, and $V' \subset \psi^{-1}(0)$.

Now let $\alpha : M \to \mathbb{R}$ be a smooth function identically equal to 1 in a neighborhood of V with its support $\operatorname{supp}(\alpha) = \operatorname{cl}\{ x \in M \mid \alpha(x) \neq 0\}$

contained in U, and let $\beta : M \to \mathbb{R}$ be a smooth function such that $\beta^{-1}(0) = V$. Choose $t = (t_1, \ldots, t_{k-1}) \in \mathbb{R}^{k-1}$ such that $0 \in \mathbb{R}^{k-1}$ is a regular value of the mapping

$$\Phi : M \to \mathbb{R}^{k-1}, \quad \text{where} \quad \Phi = (\varphi_1, \ldots, \varphi_{k-1}),$$
$$\varphi_i = \alpha\psi_i + t_i\beta, \quad i = 1, \ldots, k-1.$$

Then $W = \Phi^{-1}(0)$ is a smooth submanifold in M, containing V. Since $Y = \partial V$ is a submanifold of W of codimension 1, there exists a smooth function $f : W \to \mathbb{R}$ such that 0 is a regular value of f and $f^{-1}(0) = Y$.

Let $\varphi_k : M \to \mathbb{R}$ be a smooth extension of the function f, such that $0 \in \mathbb{R}^{k-1}$ is a regular value of $\varphi = (\varphi_1, \ldots \varphi_k)$. Then $\varphi^{-1}(0) = Y$ and the mapping φ has the required properties. $\square$

We now specify what it means for a vector bundle to be algebraic, and to admit an *algebraic structure*, cf. [3, Ch. 4].

Definition 1.4.8 *Let X be an affine real algebraic variety. A* pre-algebraic *vector bundle of rank n over X is a triple $\xi = (E, \pi, X)$, where*

(i) E is a real algebraic variety (not necessarily affine) and $\pi : E \to X$ is a regular mapping,

(ii) for each $x \in X$, the fibre $\pi^{-1}(x)$ is an n-dimensional $\mathbb{R}$-vector space,

(iii) there exists a finite covering $\{U_i\}_{i\in I}$ of X by Zariski open sets and for each $i \in I$, there exists a biregular isomorphism

$\varphi_i : U_i \times \mathbb{R}^n \to \pi^{-1}(U_i)$, such that $\pi \circ \varphi_i^{-1}$ is the canonical projection $U_i \times \mathbb{R}^n \to U_i$ and such that for each $x \in U_i$, the restriction $\{x\} \times \mathbb{R}^n \to \pi^{-1}(x)$ of φ_i is an $\mathbb{R}$-linear isomorphism.

Observe that the transition functions of a pre-algebraic vector bundle are regular mappings. Let $\xi = (E, \pi, X)$ and $\eta = (E', \pi', X')$ be two pre-algebraic vector bundles. An *algebraic morphism* $\psi : \xi \to \eta$ is a regular mapping $\psi : E \to E'$ such that $\pi' \circ \psi = \pi$ and the mapping $\psi_x : \pi^{-1}(x) \to (\pi')^{-1}(x)$ is linear for each $x \in X$. The bundles ξ and η are called *algebraically isomorphic* if there exists algebraic morphisms $\psi : \xi \to \eta$ and $\varphi : \eta \to \xi$ such that $\varphi \circ \psi = \mathrm{id}_\xi$ and $\psi \circ \varphi = \mathrm{id}_\eta$.

Definition 1.4.9 *A pre-algebraic vector bundle ξ over X is said to be* algebraic *if there exists an injective algebraic morphism from ξ to a trivial bundle ε_X^n (ε_X^n denotes the trivial vector bundle $(X \times \mathbb{R}^n, \pi, X)$ where π is the canonical projection).*

Example 1.4.10 *Recall that there is a natural bijection of the Grassmannian $G_{n,k}(\mathbb{R})$ onto the set*

$$H_{n,k} = \left\{ A \in M(n, n; \mathbb{R}) \mid {}^tA = A^2 = A,\ \mathrm{trace}(A) = k \right\},$$

which sends the vector space $V \in G_{n,k}(\mathbb{R})$ to the matrix of the orthogonal projection onto V. Let

$$E_{n,k} = \{ (A, v) \in G_{n,k}(\mathbb{R}) \times \mathbb{R}^n \mid Av = v \}$$

and let $p_{n,k} : E_{n,k} \to G_{n,k}(\mathbb{R})$ be the canonical projection onto $G_{n,k}(\mathbb{R})$. Then

$$\gamma_{n,k} = (E_{n,k}, p_{n,k}, G_{n,k}(\mathbb{R}))$$

is an algebraic vector bundle of rank k, called the universal vector bundle*.*

If $\xi = (E, \pi, X)$ is a pre-algebraic vector bundle of rank k over an affine real algebraic variety X, then there exists a regular mapping $f : X \to G_{n,k}(\mathbb{R})$ for some $n \geq k$ such that ξ is algebraically isomorphic to the induced vector bundle $f^*(\gamma_{n,k})$, cf. [3, p. 300-303].

If ξ and η are algebraic vector bundles over X, then their Whitney sum $\xi \oplus \eta$ is also an algebraic vector bundle over X. It can be proved that the tangent bundle τ_V and the normal bundle ν_V of a nonsingular algebraic set V is algebraic [3, Prop. 12.1.9].

Definition 1.4.11 *A topological vector bundle ξ over an affine real algebraic variety X is said to* admit an algebraic structure*, if there exists an algebraic vector bundle η over X topologically isomorphic to ξ.*

In the following we define a *complex algebraic vector bundle defined over* $\mathbb{R}$.

Definition 1.4.12 *Let X be a complex algebraic variety defined over $\mathbb{R}$. A* complex algebraic vector bundle over $\mathbb{R}$ of rank n over X *is a triple $\xi = (E, \pi, X)$, where*

(i) E is a complex algebraic variety defined over $\mathbb{R}$ and $\pi : E \to X$ is a regular mapping defined over $\mathbb{R}$,

(ii) for each $x \in X$, the fibre $\pi^{-1}(x)$ is a $\mathbb{C}$-vector space,

(iii) for each $x \in X$ there exists a Zariski open neighborhood U of x in X, a nonnegative integer n, and a regular isomorphism

$\varphi : U \times \mathbb{C}^n \to \pi^{-1}(U)$ defined over $\mathbb{R}$ such that the restriction $\{y\} \times \mathbb{C}^n \to \pi^{-1}(y)$ of φ is a $\mathbb{C}$-linear isomorphism for each $y \in U$.

1.5 Albanese varieties

We review some facts about abelian varieties. For more details we refer to [17], [24], [25], and [30].

Let Λ be a lattice in the complex space $\mathbb{C}^n$. The quotient

$$X = \mathbb{C}^n / \Lambda$$

is called a *complex torus.* A complex torus which is also a complex projective variety is called an *abelian variety*. The group structure on an abelian variety is induced by the one on $\mathbb{C}^n$. An abelian variety X is called *simple* if X and 0 are the only abelian subvarieties of X.

The morphisms between abelian varieties are complex regular mappings which are also group homomorphisms. A special class of homomorphisms between abelian varieties is formed by the *isogenies.*

Definition 1.5.1 *Let X and Y be complex abelian varieties. A homomorphism $f : X \to Y$ is said to be an* isogeny *if f is surjective and has a finite kernel.*

The relation of isogeny defines an equivalence relation on the set of abelian varieties [24, p. 29], [25, Ch. 1, Cor. 2.7]. Every abelian variety X is isogenous to a product

$$X_1^{n_1} \times \ldots \times X_r^{n_r}$$

of simple mutually nonisogenous abelian varieties X_i. The abelian varieties X_i and the integers n_i are uniquely determined up to isogenies and permutations [24, p. 30], [25, Ch. 5, Thm. 3.7]. This is *Poincaré's reducibility theorem.* It is a consequence of the following statement:

For each abelian subvariety Y of an abelian variety X there exists an abelian subvariety Z of X such that X is isogenous to the product $Y \times Z$ [24, p. 28], [25, Ch. 5, Thm. 3.5].

In particular, an abelian variety X is simple if X is not isogenous to the product $X_1 \times X_2$ of two abelian varieties of positive dimension.

Let X be a nonsingular complex irreducible projective variety. Let $\Omega^1(X)$ be the space of holomorphic 1-forms on X and let $\dim \Omega^1(X) = n$. Denote by $\Omega^1(X)^*$ the dual space of $\Omega^1(X)$. The first homology group $H_1(X, \mathbb{Z})$ consists of homology classes $[\gamma]$ represented by piecewise smooth loops $\gamma : S^1 \to X$ and has rank $2n$. Define the homomorphism

$$\varphi : H_1(X, \mathbb{Z}) \to \Omega^1(X)^*, \qquad [\gamma] \mapsto \left\{ \omega \mapsto \int_\gamma \omega \right\}.$$

The kernel of φ is precisely the torsion group of $H_1(X, \mathbb{Z})$, and the image of φ forms a lattice $\Lambda_X \cong \mathbb{Z}^{2n}$ in $\Omega^1(X)^*$ [25, p. 321].

Definition 1.5.2 *Let X be a nonsingular complex irreducible projective variety with* $\dim \Omega^1(X) = n$, *and let* $\varphi : H_1(X, \mathbb{Z}) \to \Omega^1(X)^*$ *be the mapping as defined above. The* Albanese variety *of X is the n-dimensional complex torus*

$$\mathrm{Alb}(X) = \Omega^1(X)^* / \Lambda_X.$$

Fix a base point $x_0 \in X$, and define the mapping

$$\alpha_X : X \to \mathrm{Alb}\,(X)\,, \qquad x \mapsto \left\{ \omega \mapsto \int_{x_0}^{x} \omega \right\} \mathrm{mod}\, \Lambda_X \ ,$$

where the integral is taken along any piecewise smooth path in X joining x_0 and x. The mapping α_X is called the Albanese mapping.

The Albanese mapping is regular and the Albanese variety $\mathrm{Alb}\,(X)$ is an abelian variety, cf. [24, p. 41]. It satisfies the following universal property [24, p. 41] [25, p. 361]:

For every rational mapping $h : X \to A$ of X into an abelian variety A, there exists a unique homomorphism $H : \mathrm{Alb}\,(X) \to A$ such that $h = H \circ \alpha_X + c$, where $c = h\,(x_0)$.

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & Alb(X) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle H} \\ A & \xleftarrow[+c]{} & A \end{array}$$

If X is already an abelian variety, then α_X is an isomorphism and thus $\mathrm{Alb}\,(X) = X$. An abelian variety of dimension 1 is called an *elliptic curve* and is isomorphic to a nonsingular complex projective cubic, equipped with an appropriate group operation. Hence for any nonsingular complex cubic curve C, we have that $\mathrm{Alb}\,(C) = C$

It follows from the universal property of Albanese varieties that if X and Y are two birationally equivalent complex projective varieties, then $\mathrm{Alb}\,(X) = \mathrm{Alb}\,(Y)$. Therefore, the Albanese variety is a birational invariant.

Recall that by Hironaka's desingularization theorem [19], a nonsingular projective real algebraic variety X always has a nonsingular projective complexification $X'_{\mathbb{C}}$ which is unique up to birational isomorphism. As described above, we can attach the Albanese variety $\mathrm{Alb}\,(X'_{\mathbb{C}})$ to the set $X'_{\mathbb{C}}$. Since the Albanese variety is a birational invariant, it follows that $\mathrm{Alb}\,(X'_{\mathbb{C}})$ is independent of the choice of a nonsingular complexification of X in $\mathbb{R}^n$ and depends only on X. We shall therefore denote $\mathrm{Alb}\,(X'_{\mathbb{C}})$ simply by $\mathrm{Alb}\,(X)$. *Thus* $\mathrm{Alb}\,(X)$ *is a complex abelian variety attached to a nonsingular projective real algebraic variety* X*, and if* Y *is another nonsingular projective real algebraic variety, birationally equivalent to* X*, then* $\mathrm{Alb}\,(X) = \mathrm{Alb}\,(Y)$*.*

2 Representation of homology classes by real algebraic subvarieties

In this chapter we look at homology classes represented by smooth submanifolds and we give some results in which these classes play an important role. We also discuss homology classes represented by algebraic subvarieties and explain their connection with algebraic bordism groups.

2.1 Homology classes represented by smooth submanifolds

Let M be a smooth m-dimensional manifold and let $x \in M$. From the excision property for homology it follows that

$$H_m\left(M, M \backslash \{x\}; \mathbb{Z}/2\right) \cong \mathbb{Z}/2.$$

Let us denote by o_x^M the unique generator of $H_m\left(M, M \backslash \{x\}; \mathbb{Z}/2\right)$.

Proposition 2.1.1 *Let M be a compact smooth manifold of dimension m. For any point $x \in M$, let*

$$i^x : M \to \left(M, M \backslash \{x\}\right)$$

be the inclusion mapping and let

$$i_*^x : H_m\left(M, \mathbb{Z}/2\right) \to H_m\left(M, M \backslash \{x\}; \mathbb{Z}/2\right)$$

be the homomorphism induced by i^x. This homomorphism is injective and o_x^M belongs to its image. Furthermore, there exists a unique homology class

$$[M] \in H_m(M, \mathbb{Z}/2)$$

such that $i_*^x([M]) = o_x^M$.

The homology class $[M]$ is called the *fundamental class of M*. A proof of Proposition 2.1.1 may be found in [28, p. 273]. Let P be a topological space containing M, and let $i : M \to P$ be the inclusion mapping. If

$$i_* : H_m(M, \mathbb{Z}/2) \to H_m(P, \mathbb{Z}/2)$$

is the homomorphism induced by i, then we call

$$[M]_P = i_*([M])$$

the *homology class represented by M in P*.

Before we give two important results concerning homology classes represented by smooth submanifolds, we remind the reader of the following notion. Let A and B be two submanifolds of a manifold N. If at each point of $A \cap B$ the tangent spaces of A and B span the tangent space of N, then A and B are said to be in *general position*, or A is said to intersect B *transversely*.

Definition 2.1.2 *Let M and N be two smooth manifolds and let $f : M \to N$ be a smooth mapping. Suppose that $A \subset N$ is a smooth submanifold of N. The mapping f is said to be* transverse *to A, notated by $f \pitchfork A$, if for $x \in f^{-1}(A)$,*

$$T_{f(x)}A + df_x(T_xM) = T_{f(x)}N.$$

Theorem 2.1.3 *Let M and N be smooth manifolds and let A be a smooth submanifold of N. If $f : M \to N$ is transverse to A, then $f^{-1}(A)$ is a smooth submanifold of M. The codimension of $f^{-1}(A)$ in M is the same as that of A in N.*

A proof can be found in [20, p. 22].

Denote by $C^{\infty}_{W}(M,N)$ and $C^{\infty}_{S}(M,N)$ the set $C^{\infty}(M,N)$ of smooth mappings from M into N endowed with the weak and the strong topology, respectively [20, p. 34]. The set $C^{\infty}(M,N)$ is also said to be equipped with the C^{∞} *topology*. If M is compact, then the topology of the space $C^{\infty}_{W}(M,N)$ is the same as that of $C^{\infty}_{S}(M,N)$.

Theorem 2.1.4 *Let M and N be smooth manifolds and let A be a smooth submanifold of N. Then the set*

$$\pitchfork^{\infty}(M,N;A) = \{ f \in C^{\infty}(M,N) \mid f \pitchfork A \}$$

is residual (and therefore dense) in $C^{\infty}(M,N)$ endowed with the C^{∞} topology.

For a proof, see [20, p. 74]. The next result follows immediately from this theorem and from the fact that the set of smooth embeddings from M into N is open in $C^{\infty}_{S}(M,N)$.

Corollary 2.1.5 *Let A and M be smooth submanifolds of the smooth manifold N. Then every neighborhood of the inclusion $i : M \to N$ in $C^{\infty}_{S}(M,N)$ contains an embedding which is transverse to A.*

In the following, the mapping $D_M : H^p(M,\mathbb{Z}/2) \to H_{m-p}(M,\mathbb{Z}/2)$ denotes the Poincaré duality isomorphism. For details and the proofs of these results, we refer to Appendix A.

Theorem 2.1.6 *Let $f : M \to N$ be a smooth mapping between the smooth compact manifolds M and N. Let Q be a closed smooth submanifold of N and let $P = f^{-1}(Q)$. Assume that f is transverse to Q. Then*

$$D_M \circ f^* \circ D_N^{-1}([Q]_N) = [P]_M .$$

Corollary 2.1.7 *Let N_1 and N_2 be closed smooth submanifolds of a compact smooth manifold M. If N_1 intersects N_2 transversely, then*

$$D_M^{-1}([N_1]_M) \cup D_M^{-1}([N_2]_M) = D_M^{-1}([N_1 \cap N_2]_M) .$$

2.2 Hypersurfaces defined by sections of line bundles

Let M be a compact smooth manifold of dimension m, and let $V^1(M)$ be the set of isomorphism classes of smooth line bundles over M. We write ξ for the isomorphism class of a line bundle ξ in $V^1(M)$. If ξ and η are two line bundles with transition functions g_{ij}^1 and g_{ij}^2, respectively, then the transition functions of the tensor product $\xi \otimes \eta$ are given by $g_{ij} = g_{ij}^1 \cdot g_{ij}^2$. With the tensor product $\otimes$ the set $V^1(M)$ becomes a group. Define the mapping $w_1 : V^1(M) \to H^1(M, \mathbb{Z}/2)$ by

$$\xi \mapsto w_1(\xi) ,$$

where $w_1(\xi)$ is the first Stiefel-Whitney class of ξ (see [28] for definition and properties).

Theorem 2.2.1 *Let M be a compact smooth manifold. The mapping*

$$w_1 : V^1(M) \to H^1(M, \mathbb{Z}/2) , \quad \xi \mapsto w_1(\xi)$$

is an isomorphism.

Proof. Denote by $\mathcal{C}^*$ the sheaf of continuous $\mathbb{R}^*$-valued functions on M, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, and by $\mathcal{C}^+$ the subsheaf of continuous functions on M which are strictly positive. Let $\mathbb{Z}/2$ denote the sheaf of locally constant functions on M with values in $\mathbb{Z}/2$. Identify $\mathbb{Z}/2$ with the group $\{-1,1\}$ and define the morphism $r : \mathcal{C}^* \to \mathbb{Z}/2$ by

$$g \mapsto \frac{g}{|g|}.$$

Then we obtain the exact sequence

$$0 \longrightarrow \mathcal{C}^+ \longrightarrow \mathcal{C}^* \xrightarrow{\ r\ } \mathbb{Z}/2 \longrightarrow 0,$$

where 0 denotes the zero sheaf which associates the zero group to each open set of M. Since M is paracompact, cf. [13, p. 21], it follows that there exists a long exact sequence

$$\ldots \to \check{H}^p\left(M, \mathcal{C}^+\right) \to \check{H}^p\left(M, \mathcal{C}^*\right) \to \check{H}^p\left(M, \mathbb{Z}/2\right) \to \check{H}^{p+1}\left(M, \mathcal{C}^+\right) \to \ldots$$

of Čech cohomology groups, cf. [13, p. 191], [21, p. 33].

Since $\check{H}^p\left(M, \mathcal{C}^+\right) = 0$ for $p \geq 1$ [16, p. 174], it follows that

$$\check{H}^1\left(M, \mathcal{C}^*\right) \cong \check{H}^1\left(M, \mathbb{Z}/2\right).$$

The last Čech cohomology group is isomorphic to the singular cohomology group [12, p. 349, p. 539], hence

$$\check{H}^1\left(M, \mathcal{C}^*\right) \cong H^1\left(M, \mathbb{Z}/2\right). \tag{2.1}$$

The family of transition functions $\left(g_{ij}\right)_{i,j}$ of a line bundle ξ is a 1-cocycle representing a cohomology class in $\check{H}^1\left(M, \mathcal{C}^*\right)$. Conversely, given such a cocycle, one can construct a line bundle having this cocycle as the transition functions [15, p. 220]. Two cocycles represent the same cohomology class if and only if the corresponding line bundles are isomorphic [21, p. 41]. Hence the Čech cohomology group $\check{H}^1(M, \mathbb{Z}/2)$ is isomorphic with the group of isomorphism classes $V^1(M)$. This and (2.1) imply that

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$$

and that w_1 is an isomorphism. $\square$

Let $\xi = (E, \pi, M)$ be a smooth line bundle over a compact smooth manifold M and let O_E be the zero section of ξ. By Theorem 2.1.4, we can choose a smooth section $s : M \to E$ of ξ transverse to the zero section. The set $s^{-1}(O_E)$ is then a compact smooth hypersurface of M (or is $\emptyset$) and the homology class $[s^{-1}(O_E)]_M$ in $H_{m-1}(M, \mathbb{Z}/2)$ represented by $s^{-1}(O_E)$ in M is independent of the choice of the section s (Theorem 2.1.6).

Define the mapping

$$\varphi : V^1(M) \to H_{m-1}(M, \mathbb{Z}/2), \quad \xi \mapsto [s^{-1}(O_E)]_M,$$

where s is a section transverse to the zero section of ξ. Then it can be proved that

$$\varphi = D_M \circ w_1, \tag{2.2}$$

see [3, p. 313]. In particular, φ is an isomorphism. Thus for each homology class u in $H_{m-1}(M, \mathbb{Z}/2)$ there exists a smooth line bundle ξ with a section s transverse to the zero section such that $s^{-1}(O_E)$ is a compact smooth hypersurface of M and $u = [s^{-1}(O_E)]_M$. In particular, each $u \in H_{m-1}(M, \mathbb{Z}/2)$ can be represented by a compact hypersurface.

The line bundle ξ described above can be defined explicitly as follows.

Theorem 2.2.2 *Let M be a compact smooth manifold of dimension m and let N be a closed submanifold of M of codimension 1. Then there exists a smooth line bundle $\xi = (E, \pi, M)$ over M and a smooth section $s : M \to E$ such that $N = s^{-1}(O_E)$ and s is transverse to O_E.*

Proof. Let $x \in N$. Then there exists a neighborhood U_x of x in M and a function $\varphi_x : M \to \mathbb{R}$ such that the ideal

$$I = \{ f \in C^\infty(U_x) \mid f(y) = 0 \text{ for all } y \in U_x \cap N\}$$

is equal to $(\varphi_x) \cdot C^\infty(U_x)$. Since N is compact, there is a finite covering $\{U_{x_1}, \ldots, U_{x_s}\}$ of N with corresponding functions $\{\varphi_1, \ldots, \varphi_s\}$.

Define the functions

$$g_{ij} : U_{x_1} \cap U_{x_j} \to \mathbb{R} \backslash \{0\}, \quad g_{ij}(x) = \frac{\varphi_i(x)}{\varphi_j(x)}.$$

They are well-defined and satisfy the properties

$$g_{ii}(x) = 1 \text{ for every } x \in U_{x_i} \text{ and}$$
$$(g_{ij} \cdot g_{jk})(x) = g_{ik}(x) \text{ for every } x \in U_{x_i} \cap U_{x_j} \cap U_{x_k}.$$

It follows that the functions g_{ij} determine a smooth line bundle $\xi = (E, \pi, M)$ where

$$E = \left\{ (x, v_1, \ldots, v_s) \in M \times \mathbb{R}^s \middle| v_i = g_{ij}(x) v_j \text{ for } x \in U_{x_j},\ i = 1, \ldots, s \right\}$$

and $\pi : E \to M$ is the projection defined by $\pi(x, v_1, \ldots, v_s) = x$. Now let $s : M \to E$ be the section of ξ represented by the s-tuple $(\varphi_1, \ldots, \varphi_s)$. Then $N = s^{-1}(O_E)$ and s is transverse to the zero section O_E. $\square$

Lemma 2.2.3 *Let M be a smooth manifold of dimension m and let N be a submanifold of M of codimension 1 such that the homology class $[N]_M$ of N in $H_{m-1}(M,\mathbb{Z}/2)$ is Poincaré dual to $w_1(M)$. Then $M\backslash N$ is an orientable manifold.*

Proof. By Theorem 2.2.2, we can find a smooth line bundle $\xi = (E,\pi,M)$ and a smooth section $s : M \to E$ transverse to the zero section and such that $N = s^{-1}(O_E)$. The section s is then nowhere zero on $M\backslash N$. Therefore, the restriction $\xi|_{M\backslash N}$ is trivial.

Since $D_M(w_1(M)) = [N]_M$, it follows by equation (2.2) that the line bundle $\Lambda^m\tau_M$ is isomorphic to ξ and therefore has a section $\tilde{s}$ transverse to the zero section such that $N = \tilde{s}^{-1}(O_E)$. Following the same argument as with the bundle ξ, we find that $(\Lambda^m\tau_M)|_{M\backslash N}$ is trivial. This implies that $(\tau_M)|_{M\backslash N}$ is orientable, hence $M\backslash N$ is orientable. □

If τ_M is the tangent bundle of a smooth manifold M, then $w_i(\tau_M)$ is called the i-th Stiefel-Whitney class of M, and is denoted by $w_i(M)$. We will need the following theorem (see also [5, Lemma 4.3]).

Theorem 2.2.4 *Let M be a compact connected smooth manifold of dimension $m \geq 3$ and let G be a subgroup of $H^1(M,\mathbb{Z}/2)$ containing the first Stiefel-Whitney class $w_1(M)$ of M. Then for each $v \in H^1(M,\mathbb{Z}/2)\backslash G$ there exists an element $u \in H^{m-1}(M,\mathbb{Z}/2)$ such that*

(i) $v \cup u \neq 0$ *and* $w \cup u = 0$ *for all* $w \in G$.

(ii) $D_M(u) = [C]_M$ *in* $H_1(M,\mathbb{Z}/2)$ *for some compact connected smooth curve C in M embedded with trivial normal vector bundle.*

Proof. (i) Let $\cup$ be the cup product (see [29, p. 285])

$$\cup : H^1(M, \mathbb{Z}/2) \times H^{m-1}(M, \mathbb{Z}/2) \to H^m(M, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Then by [29, Thm. 68.2] there exists a basis $\{\alpha_1, \ldots, \alpha_k\}$ for $H^1(M, \mathbb{Z}/2)$ and a basis $\{\beta_1, \ldots, \beta_k\}$ for $H^{m-1}(M, \mathbb{Z}/2)$ such that

$$\alpha_i \cup \beta_j = \delta_{ij}\mu \text{ for all } i, j = 1, \ldots, k,$$

where μ is the generator of $\mathbb{Z}/2$ and δ_{ij} denotes Kronecker's delta symbol. This implies that the set

$$K = \{ u \in H^{m-1}(M, \mathbb{Z}/2) \mid w \cup u = 0 \text{ for all } w \in G \}$$

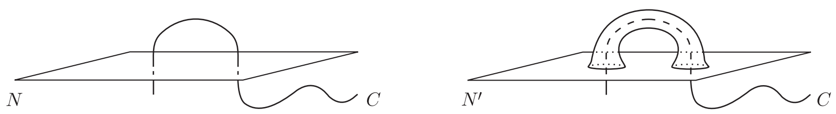
contains nontrivial elements, and that for each v in $H^1(M, \mathbb{Z}/2) \backslash G$, there exists an element u in K with $v \cup u \neq 0$.

(ii) Let C be a compact connected smooth curve in M representing the homology class in $H_1(M, \mathbb{Z}/2)$ Poincaré dual to an element $u \in K$ and let v be the homology class in $H_{m-1}(M, \mathbb{Z}/2)$ Poincaré dual to $w_1(M)$. Then Theorem 2.2.1 and 2.2.2 imply that there exists a smooth line bundle $\xi = (E, \pi, M)$ with a section s transverse to the zero section O_E such that $N = s^{-1}(O_E)$ is a compact smooth hypersurface in M and such that $v = [N]_M$. By Corollary 2.1.5, each neighborhood of the inclusion $i : N \to M$ in $C^\infty_S(M, N)$ contains an embedding, which is transverse to C. Hence we may assume that C is transverse to N.

Applying Corollary 2.1.7, we obtain

$$w_1(M) \cup u = D_M^{-1}([N]_M) \cup D_M^{-1}([C]) = D_M^{-1}([N \cap C]_M).$$

Since $w_1(M) \cup u = 0$, the cohomology class $[N \cap C]_M \in H_0(M, \mathbb{Z}/2)$ is equal to zero. Therefore, $C \cap N$ contains an even number of points. It follows that we can construct a compact smooth hypersurface N' of M by attaching a handle to N around the curve C such that $C \cap N' = \emptyset$ and such that it represents in $H_{m-1}(M, \mathbb{Z}/2)$ the same homology class as N.



Handle attached to N around C.

It follows from Lemma 2.2.3 that $M \backslash N'$ is an orientable manifold, and hence C has a trivial normal bundle in $M \backslash N'$. Thus, the normal vector bundle of C in M is trivial. □

2.3 Homology classes represented by algebraic subvarieties

Let X be a compact affine real algebraic variety of dimension n (thus possibly with singular points). By [3, Thm. 9.2.1] X can be triangulated and it can be proved that the sum of all n-simplices in X is a cycle with coefficients in $\mathbb{Z}/2$ which is not a boundary. This cycle, which we shall denote by $[X]$, represents a nonzero element of $H_n(X, \mathbb{Z}/2)$ independent of the choice of the triangulation [3, Prop. 11.3.1]. The cycle $[X]$ coincides with the notion of fundamental class in case X is nonsingular (and thus a smooth manifold) and is also called the *fundamental class* of X.

Definition 2.3.1 *Let X be a compact affine real algebraic variety of* *dimension n and let Z be a k-dimensional Zariski closed subset of X. Let*

$$i_* : H_k(Z, \mathbb{Z}/2) \to H_k(X, \mathbb{Z}/2)$$

be the homomorphism induced by the inclusion mapping $i : Z \to X$. The element $i_*([Z]) \in H_k(X, \mathbb{Z}/2)$ is called the homology class of X represented by Z *and we denote this element by $[Z]_X$.*

We denote by $H_k^{\mathrm{alg}}(X, \mathbb{Z}/2)$ the subgroup of $H_k(X, \mathbb{Z}/2)$ consisting of all homology classes represented by Zariski closed k-dimensional subsets of X. Set

$$H_*^{\mathrm{alg}}(X, \mathbb{Z}/2) = \bigoplus_{k \geq 0} H_k^{\mathrm{alg}}(X, \mathbb{Z}/2).$$

If $H_*^{\mathrm{alg}}(X, \mathbb{Z}/2) = H_*(X, \mathbb{Z}/2)$, then the homology of X is said to be *totally algebraic.*

For cohomology groups we set

$$H^*_{\mathrm{alg}}(X, \mathbb{Z}/2) = \bigoplus_{k \geq 0} H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$$

where $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2) = D_X^{-1}(H_{n-k}^{\mathrm{alg}}(X, \mathbb{Z}/2))$, and D_X is the Poincaré duality isomorphism.

Example 2.3.2 *The homology of the Grassmannians $G_{n,k}(\mathbb{R})$ is totally algebraic. In particular, the homology of the projective space $\mathbb{R}P^n$ is totally algebraic [3, p. 272].*

Theorem 2.3.3 *Let X be an affine nonsingular real algebraic variety and let A be a Zariski closed subset of X of pure codimension* 1 *(i.e. each* *irreducible component of A is of codimension* 1*). Then there exists an algebraic $\mathbb{R}$-line bundle $\xi = (E, \pi, X)$ and an algebraic section $s : X \to E$ such that $A = s^{-1}(O_E)$ and s is transverse to O_E at each point of* $\operatorname{Reg}(A)$*.*

Proof. Let

$$I = I(A) = \{ f \in \mathcal{R}(X) \mid f(x) = 0 \text{ for all } x \in A \}.$$

Since $\mathcal{R}(X)$ is a Noetherian ring, we can choose finitely many generators $f_1, \ldots, f_n$ of I. Define a subset U_j of X by

$$U_j = \{ x \in X \mid I\mathcal{R}_{X,x} = (f_j)\,\mathcal{R}_{X,x} \},$$

where $\mathcal{R}_{X,x}$ denotes the local ring of X, i.e. the ring of germs (for the Zariski topology) of regular functions at $x \in X$. One sees that the set U_j is Zariski open in X. Since the $\mathcal{R}_{X,x}$ are factorial rings, it follows that $\{U_1, \ldots, U_n\}$ is a cover of X, and f_j is transverse to $0 \in \mathbb{R}$ at each point of $U_j \cap \operatorname{Reg}(A)$. Furthermore, $f_i = h_{ij} f_j$ on U_j for some regular functions h_{ij} in $\mathcal{R}(U_j)$. Note that $h_{ii}(x) = 1$ on U_i and $h_{ij} h_{jk} = h_{ik}$ on $U_j \cap U_k$. Let

$$E = \{ (x, v_1, \ldots, v_n) \in X \times \mathbb{R}^n \mid v_i = h_{ij}(x)\, v_j \text{ for } x \in U_j,\ 1 \le i \le n \}$$

and let $\pi : E \to X$ be the mapping defined by $\pi((x, v_1, \ldots, v_n)) = x$. Then $\xi = (E, \pi, X)$ is an algebraic subbundle of the trivial bundle ε_X^n. By construction, ξ is a line bundle. We define an algebraic section $s : X \to E$ by $s(x) = (x, f_1(x), \ldots, f_n(x))$. Then $A = s^{-1}(O_E)$ and s is transverse to O_E at each point of $\operatorname{Reg}(A)$. $\square$

We can now prove the following useful result, cf. [3, Remark 12.4.7].

Theorem 2.3.4 *Let X and Y be compact affine nonsingular real algebraic varieties and let $f : X \to Y$ be a regular mapping. Then*

$$f^*(H^1_{\text{alg}}(Y, \mathbb{Z}/2)) \subset H^1_{\text{alg}}(X, \mathbb{Z}/2)\,.$$

Proof. Let $u \in H^1_{\text{alg}}(Y, \mathbb{Z}/2)$. Then $D_Y(u) \in H^{\text{alg}}_{d-1}(Y, \mathbb{Z}/2)$, that is, u is Poincaré dual to $[B]_X$ for some algebraic subvariety B of codimension 1 in X. By Theorem 2.3.3, there exists an algebraic line bundle $\xi = (E, \pi, Y)$ on Y and an algebraic section $s : Y \to E$ such that $B = s^{-1}(O_E)$ and s is transverse to O_E. Formula (2.2) implies that $D(w_1(\xi)) = [B]_Y$, therefore, $u = w_1(\xi)$.

Now $f^*(w_1(\xi)) = w_1(f^*(\xi))$, and $f^*(\xi) = (E', \pi', X)$ is an algebraic line bundle over X [3, Prop. 12.1.8 (ii)]. Let σ be a smooth section of $f^*(\xi)$, transverse to the zero section $O_{E'}$. Applying Theorem 3.1.6 (with $A = \emptyset$)†, σ can be approximated in the C^∞ topology of $C^\infty(X, E')$ by an algebraic section $s' : X \to E'$ of $f^*(\xi)$, transverse to $O_{E'}$. Then $A = (s')^{-1}(O_{E'})$ is a nonsingular Zariski closed subvariety of X and $D_Y(w_1(f^*(\xi))) = [A]_Y$. It follows that $w_1(f^*(\xi)) \in H^1_{\text{alg}}(Y, \mathbb{Z}/2)$, which proves the theorem. $\square$

2.4 Algebraic bordism groups

Consider two compact smooth m-dimensional manifolds M_0 and M_1. They are called *cobordant* if their disjoint union $M_0 \sqcup M_1$ is the

†Theorem 3.1.6 is an approximation theorem proved in the next chapter.

boundary of a compact smooth $(m+1)$-dimensional manifold with boundary. The manifolds M_0 and M_1 are said to belong to the same *cobordism class*. See [14] and [36] for more on cobordism theory.

Definition 2.4.1 *Let Y be a smooth manifold and let $f_0 : M_0 \to Y$ and $f_1 : M_1 \to Y$ be smooth mappings of compact smooth m-dimensional manifolds M_0 and M_1 into Y. The pairs (M_0, f_0) and (M_1, f_1) are said to belong to the same* bordism class *if there exists a compact smooth manifold P with boundary and a smooth mapping $f : P \to Y$ such that $\partial P = M_0 \sqcup M_1$, $f|_{M_0} = f_0$, and $f|_{M_1} = f_1$.*

The set of all bordism classes of pairs (M, f), where M is a compact smooth m-dimensional manifold and $f : M \to Y$ is a smooth mapping into a smooth manifold Y, is denoted by $\mathfrak{N}_m(Y)$. The sum of two classes represented by (M_0, f_0) and (M_1, f_1) is the bordism class of the pair $(M_0 \sqcup M_1, f_0 \sqcup f_1)$, where $f_0 \sqcup f_1 : M_0 \sqcup M_1 \to Y$ is the smooth mapping defined by $f_0 \sqcup f_1|_{M_i} = f_i$ for $i = 0, 1$. The zero element of $\mathfrak{N}_m(Y)$ is represented by the pair (M_0, f_0), where M_0 is the boundary of a compact smooth manifold with boundary and f_0 extends to a smooth mapping $f : P \to Y$. The set $\mathfrak{N}_m(Y)$ is called the *unoriented bordism group of Y* and it is clear that every element of the group $\mathfrak{N}_m(Y)$ is of order 2. The direct sum

$$\mathfrak{N}_*(Y) = \bigoplus_{m=0}^{\infty} \mathfrak{N}_m(Y)$$

is a graded $\mathfrak{N}_*$-module. The product of the classes represented by M in $\mathfrak{N}_m$ and by (N, f) in $\mathfrak{N}_n(Y)$ is the bordism class in $\mathfrak{N}_{m+n}(Y)$ represented by $(M \times N, \bar{f})$, where $\bar{f} : M \times N \to Y$ is defined by $\bar{f}(x, y) = f(y)$ for all $(x, y) \in M \times N$.

Definition 2.4.2 *Let Y be an affine nonsingular real algebraic variety. A bordism class in $\mathfrak{N}_*(Y)$ is called* algebraic *if it can be represented by a pair (V,f), where V is a compact affine nonsingular real algebraic variety and $f:V\to Y$ is a regular mapping.*

Denote by $\mathfrak{N}_*^{\text{alg}}(Y)$ the subset of $\mathfrak{N}_*(Y)$ consisting of all algebraic bordism classes. It is a subgroup of $\mathfrak{N}_*(Y)$ and is called the *algebraic bordism group of Y*. The connection between algebraic bordism groups and algebraic homology groups is expressed in Theorem 2.4.3 (cf. [2, Lemma 2.7.1]).

Theorem 2.4.3 *Let Y be an affine nonsingular compact real algebraic variety. Then the following conditions are equivalent:*

(i) $\mathfrak{N}_*^{\text{alg}}(Y)=\mathfrak{N}_*(Y)$,

(ii) $H_*^{\text{alg}}(Y,\mathbb{Z}/2)=H_*(Y,\mathbb{Z}/2)$.

Proof. $(i)\Rightarrow(ii)$ Consider the canonical homomorphism

$$\mathfrak{N}_*(Y)\to H_*(Y,\mathbb{Z}/2)$$

which associates to the bordism class represented by (M,f) the homology class $f_*([M])$, where $[M]$ denotes the fundamental class of M. By [36, p. 108] this homomorphism is surjective. Since

$$f_*(H_*^{\text{alg}}(M,\mathbb{Z}/2))\subset H_*^{\text{alg}}(Y,\mathbb{Z}/2),$$

cf. Appendix B, this implies (ii).

$(ii)\Rightarrow(i)$ Suppose that $H_*^{\text{alg}}(Y,\mathbb{Z}/2)=H_*(Y,\mathbb{Z}/2)$. Then we can find pairs (V_i, f_i), $i = 1, \ldots, n$, such that the V_i are compact affine real algebraic varieties, $f_i : V_i \to Y$ are regular mappings, and the $(f_i)_* ([V_i])$ generate $H_* (Y, \mathbb{Z}/2)$, where $[V_i]$ is the fundamental class of V_i (cf. Section 2.2.1). Hironaka's desingularization theorem [19] allows one to assume that the V_i are nonsingular. Let $W_1, W_2, \ldots$ be compact affine nonsingular real algebraic varieties whose cobordism class in $\mathfrak{N}_*$ generate $\mathfrak{N}_*$ as an abelian group. Let $\pi_{ij} : V_i \times W_j \to V_i$ be the canonical projection. It follows from [36, p. 108] that $\mathfrak{N}_* (Y)$ is generated as an abelian group by the bordism classes of $(V_i \times W_j, f_i \circ \pi_{ij})$ for $i = 1, \ldots, n$, $j = 1, 2, \ldots$. Hence, $\mathfrak{N}_*^{\mathrm{alg}} (Y) = \mathfrak{N}_* (Y)$. $\square$

Together with Künneth's formula, Theorem 2.4.3 implies the following.

Theorem 2.4.4 *Let Y_1 and Y_2 be compact affine nonsingular real algebraic varieties. Then the following conditions are equivalent:*

(i) $\mathfrak{N}_*^{\mathrm{alg}} (Y_i) = \mathfrak{N}_* (Y_i)$ *for* $i = 1, 2$,

(ii) $\mathfrak{N}_*^{\mathrm{alg}} (Y_1 \times Y_2) = \mathfrak{N}_* (Y_1 \times Y_2)$.

Example 2.4.5 *Since* $H_*^{\mathrm{alg}} (G_{n,k} (\mathbb{R}), \mathbb{Z}/2) = H_* (G_{n,k} (\mathbb{R}), \mathbb{Z}/2)$, *Theorem 2.4.3 implies that* $\mathfrak{N}_*^{\mathrm{alg}} (G_{n,k} (\mathbb{R})) = \mathfrak{N}_* (G_{n,k} (\mathbb{R}))$.

3 Algebraic approximations of smooth manifolds

Several approximation theorems are discussed in this chapter. In particular, we look at approximation of smooth manifolds by algebraic subvarieties. The chapter ends with Theorem 3.2.8, an approximation theorem which will be of great importance in the next chapter.

3.1 Approximation theorems

We begin this section with some basic approximation theorems.

Theorem 3.1.1 *Let M and N be two smooth manifolds such that $\dim N \geq 2\dim M + 1$. If M is compact, then the set of embeddings from M into N is dense in $C^\infty(M, N)$.*

For a proof see [20, p. 55]. Many approximation theorems are based on the approximation theorem of Stone-Weierstrass, which we formulate next. A proof can be found in [31, p. 33].

Theorem 3.1.2 (Stone Weierstrass) *Let $K \subset \mathbb{R}^n$ be a compact subset and let $f : U \to \mathbb{R}$ be a C^∞ function defined on an open neighborhood U of K. Let $\varepsilon > 0$, and s be a positive integer. Then there exists a poly-*

nomial $g \in \mathbb{R}[X_1, \ldots, X_n]$ such that

$$\left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} (f - g)(x) \right| < \varepsilon$$

for each $x \in K$ and each $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq s$.

Corollary 3.1.3 *Let X be a compact nonsingular affine real algebraic variety. Then the set $\mathcal{R}(X)$ of regular functions on X is dense in the set $C^{\infty}(X)$.*

Proof. For an algebraic set $X \subset \mathbb{R}^n$, the mapping

$$C^{\infty}(\mathbb{R}^n) \to C^{\infty}(X), \quad \varphi \mapsto \varphi|_X$$

is surjective. Then the corollary follows from the Stone-Weierstrass theorem. □

Definition 3.1.4 *Let X be a nonsingular affine real algebraic variety. A closed subset A of X is said to be* quasi-regular *if the ideal of $C^{\infty}(X)$ of all smooth functions vanishing on A is generated by regular functions.*

Clearly, if the set A is quasi-regular, then A is Zariski closed. If A is a nonsingular Zariski closed subset of X, then A is quasi-regular.

Proposition 3.1.5 *Let X be a compact nonsingular affine real algebraic variety and let A be a quasi-regular Zariski closed subset of X. Let $f : X \to \mathbb{R}$ be a smooth function such that $f|_A : A \to \mathbb{R}$ is regular. Then given a neighborhood $\mathcal{V}$ of f in $C^{\infty}(X)$, there exists a regular function $g : X \to \mathbb{R}$ such that $g \in \mathcal{V}$ and $g|_A = f|_A$.*

Proof. Let $\varphi_1, \ldots, \varphi_k$ be regular functions in $\mathcal{R}(X)$ generating the ideal of the ring $C^\infty(X)$ of all smooth functions vanishing on A. Take a regular function $h : X \to \mathbb{R}$ such that $f|_A = h|_A$ and write

$$f - h = \sum_{i=1}^{k} \alpha_i \varphi_i \ ,$$

for some functions $\alpha_1, \ldots, \alpha_k$ in $C^\infty(X)$.

Now take

$$g = h + \sum_{i=1}^{k} \beta_i \varphi_i \ ,$$

where each β_i is a regular function in $\mathcal{R}(X)$ approximating α_i in $C^\infty(X)$ sufficiently close (Corollary 3.1.3). Then $g|_A = f|_A$ and

$$f - g = \sum_{i=1}^{k} (\alpha_i - \beta_i) \varphi_i$$

so that g is close to f in $C^\infty(X)$. □

Let X be an affine nonsingular real algebraic variety and let $\xi = (E, \pi, X)$ be an algebraic vector bundle over X. We denote by $\Gamma^\infty(\xi)$ the space of all smooth sections of ξ, endowed with the topology induced from $C^\infty(X, E)$.

Theorem 3.1.6 *Let X be an affine nonsingular real algebraic variety and let ξ be an algebraic vector bundle over X. Let A be a Zariski closed subset of X and let $\sigma : X \to \xi$ be a smooth section of ξ such that $\sigma|_A : A \to \xi|_A$ is algebraic. If A is quasi-regular, then for each neighborhood $\mathcal{V}$ of σ in $\Gamma^\infty(\xi)$ there exists an algebraic section $s : X \to \xi$ such that $s \in \mathcal{V}$ and $s|_A = \sigma|_A$.*

Proof. By [3, Thm. 12.1.7 (ii)], there exists an algebraic vector bundle η over X such that $\xi \oplus \eta$ is algebraically isomorphic to the trivial vector bundle ε_X^n, for some positive integer n. Thus it suffices to prove Theorem 3.1.6 with $\xi = \varepsilon_X^n$. Then the conclusion follows from Proposition 3.1.5.

$\square$

Proposition 3.1.7 *Let $\xi = (E, \pi, X)$ be a topological vector bundle over a compact affine real algebraic variety X. If there exists algebraic vector bundles η and ζ over X such that $\xi \oplus \eta$ is topologically isomorphic to ζ, then ξ admits an algebraic structure.*

Proof. Let $k : \xi \to \zeta$ and $j : \eta \to \zeta$ be topological morphism such that the mapping $(k, j) : \xi \oplus \eta \to \zeta$ defined by $(u, v) \longmapsto k(u) + j(v)$ is a topological isomorphism. Then j determines a continuous section of the algebraic vector bundle $\mathrm{Hom}(\eta, \zeta)$. By Theorem 3.1.6, this section can be approximated close enough by an algebraic section. Thus we obtain an algebraic morphism $j' : \eta \to \zeta$ such that $(k, j') : \xi \oplus \eta \to \zeta$ is an isomorphism. Then ξ is topologically isomorphic to the vector bundle $\mathrm{Coker}(j')$, which is algebraic. $\square$

The following result from [3, p. 352] concerns the approximation of continuous or smooth mappings into Grassmannians by regular mappings.

Theorem 3.1.8 *Let X be a compact affine real algebraic (resp. nonsingular) variety and let $f : X \to G_{n,p}(\mathbb{R})$ be a continuous (resp. smooth) mapping. Then the following properties are equivalent:*

(*i*) *The induced* $\mathbb{R}$*-vector bundle* $f^*(\gamma_{n,p})$ *is topologically isomorphic to an algebraic* $\mathbb{R}$*-vector bundle.*

(*ii*) *The mapping* f *can be approximated, in the* C^0 *(resp.* C^∞*) topology, by regular mappings* $X \to G_{n,p}(\mathbb{R})$*.*

(*iii*) *The mapping* f *is homotopic to a regular mapping* $X \to G_{n,p}(\mathbb{R})$*.*

Proposition 3.1.9 *Let* K *be a compact smooth submanifold of* $\mathbb{R}^d$ *(possibly with boundary) and let* V *be an algebraic subset of* $\mathbb{R}^d$ *such that* $V \subset K$*. Let* $f : K \to G_{n,p}(\mathbb{R})$ *be a smooth mapping such that* $(f^*\gamma_{n,p})|_V$ *admits an algebraic structure. Then for each neighborhood* $\mathcal{V}$ *of* f *in* $C^\infty(K, G_{n,p}(\mathbb{R}))$ *there exists a mapping* $g : K \to G_{n,p}(\mathbb{R})$ *such that* $g \in \mathcal{V}$ *and the restriction* $g|_V : V \to G_{n,p}(\mathbb{R})$ *is regular.*

Proof. By assumption and by Theorem 3.1.8, there exists a regular mapping $r : V \to G_{n,p}(\mathbb{R})$ closely approximating $f|_V$ in $C^\infty(K, G_{n,p}(\mathbb{R}))$. Let $g : K \to G_{n,p}(\mathbb{R})$ be a smooth mapping such that $g|_V = r$ (to construct g, consider $G_{n,p}(\mathbb{R})$ as a Zariski closed subset of $\mathbb{R}^N$ for some N and use a smooth retraction $\rho : T \to G$ of some neighborhood T of G in $\mathbb{R}^N$). If r is close enough to $f|_V$, then g can be chosen in $\mathcal{V}$. $\square$

3.2 Algebraic approximation of smooth submanifolds

Let X be an affine nonsingular real algebraic variety and let M be a compact smooth submanifold of X. Let $\mathcal{U}$ be a neighborhood of the inclusion mapping $i : M \to X$ in $\mathrm{Emb}^\infty(M, X)$, the set of all smooth embeddings from M into X. An *algebraic approximation* of the manifold

M in X is a Zariski closed nonsingular subset V of X which is the image of a smooth embedding in $\mathcal{U}$. This definition can be extended as follows.

Definition 3.2.1 *Let X and Y be affine nonsingular real algebraic varieties and let A be a Zariski closed subset of X. Let M be a compact smooth submanifold of X containing A, and let $f : M \to Y$ be a smooth mapping. Assume that $f|_A : A \to Y$ is a regular mapping. Let $\mathcal{U}$ be a neighborhood of the inclusion mapping $i : M \to X$ in* $\mathrm{Emb}^\infty(M, X)$ *and let $\mathcal{V}$ be a neighborhood of f in $C^\infty(M, Y)$.*

Given a Zariski closed nonsingular subset V of X and a regular mapping $g : V \to Y$, the pair (V, g) is called an algebraic $(\mathcal{U}, \mathcal{V})$-approximation of (M, f) in X relative to A, *if there exists a smooth embedding $e : M \to X$ in $\mathcal{U}$ such that*

$$V = e(M),\ e(x) = x \text{ for all } x \in A, \text{ and } g \circ e \in \mathcal{V}.$$

The pair (M, f) is said to admit *an algebraic approximation in X relative to A if for each $\mathcal{U}$ and $\mathcal{V}$ as above, there exists an algebraic $(\mathcal{U}, \mathcal{V})$-approximation of (M, f) in X relative to A.*

For any nonnegative integer k, we can regard X as a subset of $X \times \mathbb{R}^k$ identifying X and $X \times \{0\}$. The variety X can also be regarded as a subset of $X \times \mathbb{R}P^k$, X being identified with $X \times \{(1 : 0 : \ldots : 0)\}$. Consequently, we can consider a submanifold M of X as a submanifold of $X \times \mathbb{R}^k$ (or $X \times \mathbb{R}P^k$) and A as a subvariety $X \times \mathbb{R}^k$ (or $X \times \mathbb{R}P^k$), so that we can speak of algebraic approximations of (M, f) in $X \times \mathbb{R}^k$ (or $X \times \mathbb{R}P^k$) relative to A.

Theorem 3.1.6 implies the following useful algebraic approximation theorem.

Corollary 3.2.2 *Let X be a compact affine nonsingular real algebraic variety of dimension n and let D be a compact smooth hypersurface of X. Let A be a quasi-regular Zariski closed subset of X contained in D. Then the following properties are equivalent:*

(i) *D admits an algebraic approximation in X relative to A.*

(ii) *The homology class $[D]_X$ is in $H^{alg}_{n-1}(X, \mathbb{Z}/2)$.*

Proof. $(i) \Rightarrow (ii)$ Follows directly.
$(ii) \Rightarrow (i)$ By Theorem 2.3.3 there exists an algebraic $\mathbb{R}$–line bundle $\xi = (E, \pi, X)$ over X and an algebraic section s of ξ, transverse to the zero section of ξ, such that $D = s^{-1}(O_E)$. By Theorem 3.1.6, we can choose an algebraic section σ of ξ approximating s arbitrarily close in the C^∞ topology and vanishing on A. Then $\sigma^{-1}(O_E)$ is an algebraic approximation of D in X relative to A. $\square$

Remark 3.2.3 *By definition, (i) of Corollary 3.2.2 means that for each neighborhood $\mathcal{U}$ of the inclusion mapping $i : D \to X$, there exists a Zariski closed nonsingular subset V of X and a smooth embedding e in $\mathcal{U}$ such that $e(D) = V$ and $e(x) = x$ for all $x \in A$. If $\mathcal{U}$ is chosen small enough, the submanifolds D and V are diffeotopic in X, cf. [3, p. 318].*

The main theorem of this section will be crucial for our investigations. It is contained in unpublished lecture notes by Bochnak and Kucharz [10]. Before formulating and showing the proof of this theorem, we need some more definitions and technical results. Theorem 3.2.4 is proved in [1, p. 51], Proposition 3.2.5 in [3, p. 375], Proposition 3.2.6 in [3, p. 69], and Proposition 3.2.7 in [3, p. 373].

Theorem 3.2.4 (Thom's isotopy theorem) *Let N be a compact smooth submanifold of $\mathbb{R}^k$ with possible nonempty boundary.*
Let $f : N \to P$ be a smooth mapping on N into a smooth manifold P such that f is transverse to a smooth submanifold Q of P and $f^{-1}(Q) \cap \partial N = \emptyset$. Let (T, ρ) be a closed tubular neighborhood of $f^{-1}(Q)$ in N such that $T \cap \partial N = \emptyset$. Then for every neighborhood $\mathcal{U}$ of the inclusion mapping $f^{-1}(Q) \to T$ in $\mathrm{Emb}^{\infty}\left(f^{-1}(Q), T\right)$, there exists a neighborhood $\mathcal{V}$ of $f : N \to P$ in $C^{\infty}(N, P)$ such that the following holds:

- (i) *Each mapping $g \in \mathcal{V}$ is transverse to Q,*
- (ii) $g^{-1}(Q) \subset T \backslash \partial T$,
- (iii) *the mapping $\bar{\rho} = \rho|_{g^{-1}(Q)} : g^{-1}(Q) \to f^{-1}(Q)$ is a smooth diffeomorphism,*
- (iv) *the smooth embedding $e : f^{-1}(Q) \to T$ defined by $e(u) = (\bar{\rho})^{-1}(u)$ for all $u \in f^{-1}(Q)$, belongs to $\mathcal{U}$.*

In particular, if $\mathcal{U}$ is sufficiently small, then the smooth subvarieties $f^{-1}(Q)$ and $g^{-1}(Q)$ of N are diffeotopic.

Proposition 3.2.5 *Let X and Y be nonsingular algebraic subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Let A be a quasi-regular algebraic subset of X. Assume that A is contained in a compact smooth submanifold (with* *boundary) K of X, with $\dim K = \dim X = k$. Let $f : K \to Y$ be a smooth mapping such that $f|_A$ is regular. Let $\mathcal{V}$ be a neighborhood of f in $C^\infty(K, Y)$ and let $\mathcal{W}$ be a neighborhood of the constant mapping $K \to \mathbb{R}^p$, $x \mapsto 0$, in $C^\infty(K, \mathbb{R}^p)$. Then there exist an algebraic subset Z of $X \times \mathbb{R}^p$, $\dim Z = k$, $A = A \times \{0\} \subset Z$, a positive real number ε, a smooth mapping $\beta : K \to \mathbb{R}^p$, and a regular mapping $r : Z \to Y$, such that*

(i) $\beta \in \mathcal{W}$ *and* $\beta|_A = 0$,

(ii) the set $Z_0 = \{ (x, u) \in Z \mid x \in K, \|u\| < \varepsilon \}$ *is a compact submanifold (with boundary) in* $\operatorname{Reg}(Z)$ *with* $\dim Z_0 = k$. *Moreover, the mapping* $\sigma : K \to X \times \mathbb{R}^p$, *defined by* $\sigma(x) = (x, \beta(x))$ *is a smooth diffeomorphism from* K *into* Z_0,

(iii) $r \circ \sigma \in \mathcal{V}$ *and* $r(x, 0) = f(x)$ *for all* $x \in A$.

Proposition 3.2.6 *Let V and W be nonsingular affine real algebraic varieties. Assume that $\dim V = \dim W$, $V \subset W$, $V \neq W$. Then $W \backslash V$ is a nonsingular affine real algebraic variety.*

Proposition 3.2.7 *Let $W \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^n$ be algebraic sets, with Z nonsingular of dimension s and let $f : W \to \mathbb{R}^n$ be a regular mapping. Let $x \in f^{-1}(Z)$ be a point in W which is nonsingular in dimension r, and assume f is transverse to Z at x. Then x is a point of the algebraic set $f^{-1}(Z)$, nonsingular in dimension $r + s - n$.*

Now we are ready to prove the main result of this section.

Theorem 3.2.8 *Let X and Y be affine nonsingular real algebraic varieties. Let A be a Zariski closed quasi-regular subset of X. Let M be a compact smooth submanifold of X containing A and let $i : M \to X$ be the inclusion mapping. Let $f : M \to Y$ be a smooth mapping whose restriction $f|_A : A \to Y$ is regular. Assume that the restriction $\tau_M|_A$ of the tangent bundle τ_M to A admits an algebraic structure and the bordism class of $(M, (i, f))$ in $\mathfrak{N}_* (X \times Y)$ is algebraic. Then there exists a nonnegative integer k such that (M, f) admits an algebraic approximation in $X \times \mathbb{R}P^k$ relative to A.*

Proof. The proof consists of five parts.

PART 1. We prove the following claim.

CLAIM 1. There exists a nonnegative integer ℓ, a Zariski closed nonsingular subset W of $X \times \mathbb{R}^\ell$, a regular mapping (s_X, s_Y) from W into $X \times Y$, and a pair $(P, (h_X, h_Y))$ such that the following conditions are satisfied:

(1) P is a compact smooth submanifold (with boundary) in $X \times \mathbb{R}^{\ell+1} = X \times \mathbb{R}^\ell \times \mathbb{R}$.

(2) $P \subset X \times \mathbb{R}^\ell \times [0, \infty)$ and $\partial P = M \cup W$.

(3) $(h_X, h_Y) : P \to X \times Y$ is a smooth mapping such that

$$(h_X, h_Y)|_M = (i, f) \quad \text{and} \quad (h_X, h_Y)|_W = (s_X, s_Y) .$$

(4) $P \cap X \times \mathbb{R}^\ell \times [0, 1) = (M \times [0, 1)) \cup (W \times [0, 1))$, and $P \cap X \times \mathbb{R}^\ell = M \cup W$.

Proof of Claim 1. By assumption, the bordism class of $(M,(i,f))$ in $\mathfrak{N}_*(X\times Y)$ is algebraic. Thus there exists a compact affine nonsingular real algebraic variety B and a regular mapping $b: B\to X\times Y$ such that $(M,(i,f))$ and (B,b) belong to the same bordism class, that is, there exists a compact smooth manifold (with boundary) Q and a smooth mapping $t: Q\to X\times Y$ such that

$$\partial Q = M\sqcup B,\ \ t|_M=(i,f)\,,\text{ and }\ t|_B=b.$$

We may assume that Q is a smooth submanifold of $\mathbb{R}^\ell$ for some ℓ, and that B is Zariski closed in $\mathbb{R}P^\ell$.

Let $a: Q\to\mathbb{R}^\ell$ be a smooth mapping such that $a(x)=0$ for $x\in M$ and $a(x)=x$ for $x\in B$. Write $t=(t_X,t_Y)$ and $b=(b_X,b_Y)$ and consider the smooth mapping

$$(t_X,a): Q\to X\times\mathbb{R}^\ell.$$

Then $(t_X,a)(x)=(i(x),0)$ for $x\in M$ and $(t_X,a)(x)=(b_X(x),x)$ for $x\in B$. Since the restriction

$$(t_X,a)|_{M\cup B}: M\cup B\to X\times\mathbb{R}^\ell$$

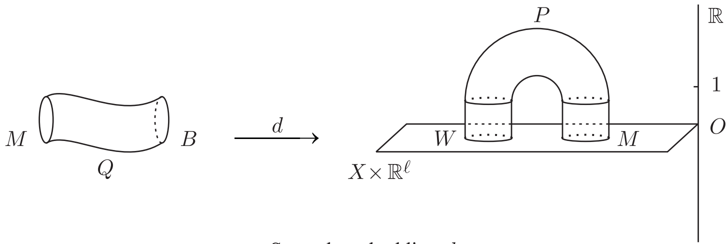
is a smooth embedding, we may assume, increasing ℓ if necessary, that there is a smooth embedding

$$d: Q\to X\times\mathbb{R}^\ell\times\mathbb{R}$$

with

$$d|_{M\cup B}=(t_X,a)|_{M\cup B}\,,\quad d(Q)\cap\left(X\times\mathbb{R}^\ell\right)=M\cup W,\ \text{ and}$$
$$d(Q)\cap\left(X\times\mathbb{R}^\ell\times[0,1)\right)=(M\times[0,1))\cup(W\times[0,1))\,,$$

where $W = \{(b_X(x), x) \mid x \in B\} = d(B)$.



Smooth embedding d.

Then W is a nonsingular Zariski closed subset of $X \times \mathbb{R}^{\ell}$. Taking

$$P = d(Q), \quad (h_X, h_Y) = (t_X, t_Y) \circ d^{-1} : P \to X \times Y,$$
$$\text{and} \quad (s_X, s_Y) = (b_X, b_Y) \circ (d|_B)^{-1} : W \to X \times Y,$$

conditions (1)-(4) are satisfied.

PART 2. Consider the mapping $\alpha : X \times \mathbb{R}^{\ell} \times \mathbb{R} \to X \times \mathbb{R}^{\ell} \times \mathbb{R}$ defined by

$$\alpha(x, u, v) = (x, u, -v) \quad \text{for all} \quad (x, u, v) \in X \times \mathbb{R}^{\ell} \times \mathbb{R}.$$

It follows from (1), (2), and (4) that the set

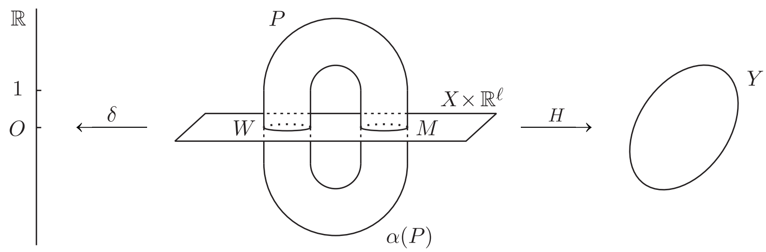
$$N = P \cup \alpha(P)$$

is a compact smooth submanifold of $X \times \mathbb{R}^{\ell+1}$. Using a partition of unity one can construct a smooth extension $H : N \to Y$ of the mapping $h_Y : P \to Y$. Note that the "height function" $\delta : N \to \mathbb{R}$ defined by

$$\delta(x, u, v) = v$$

is smooth, and

(5) $\partial P = \delta^{-1}(0)$ and δ is transverse to 0 in $\mathbb{R}$.



Submanifold $N = P \cup \alpha(P)$ with mappings H and δ.

We may assume that X is an algebraic subset of $\mathbb{R}^n$ for some n. Let (T, ρ) be a closed tubular neighborhood of N in $X \times \mathbb{R}^{\ell+1} \subset \mathbb{R}^n \times \mathbb{R}^{\ell+1}$ and let $\theta' : T \to \mathbb{R}^{n+\ell+1}$ be the corresponding orthogonalization mapping. Then $\theta'(t) = 0$ for $t \in N$ and $\theta'(t)$ is a vector tangent to $X \times \mathbb{R}^{\ell+1}$ at $\rho(t)$ and perpendicular to $T_{\rho(t)}N$ (cf. Thm 1.4.3). Let c be the codimension of N in $X \times \mathbb{R}^{\ell+1}$. Denote by G the Grassmannian $G_{n+\ell+1,c}(\mathbb{R})$ and by γ the universal vector bundle $\gamma_{n+\ell+1,c}(\mathbb{R})$ over G. The Grassmannian G is identified with the algebraic set

$$\left\{ L \in M_{n+\ell+1,n+\ell+1}(\mathbb{R}) \mid {}^tL = L = L^2, \mathrm{trace}(L) = c \right\},$$

by sending each c-dimensional vector space V of $\mathbb{R}^{n+\ell+1}$ to the matrix L of the orthogonal projection onto V. The total space E of the universal vector bundle γ is the set

$$E = \left\{ (L, y) \in G \times \mathbb{R}^{n+\ell+1} \middle| \ L \cdot y = y \right\}$$

(see Example 1.4.10). Let V be the vector space

$$V = \left\{ v \in \mathbb{R}^{n+\ell+1} \middle| \ v \in T_{\rho(y)}(X \times \mathbb{R}^{\ell+1}) \text{ and } v \perp T_{\rho(y)}N \right\}$$

and let $L(y)$ be the matrix of the orthogonal projection onto V (so that the image $L(y)\left(\mathbb{R}^{n+\ell+1}\right)$ is the normal space of N in $X \times \mathbb{R}^{l+1}$ at $\rho(y)$). Define the mapping $\varphi : T \to E$ by

$$\varphi(y) = \left(L(y), \theta'(y)\right) \quad \text{for all } y \in T.$$

Then φ is a smooth mapping such that $\varphi^{-1}(G) = N$ (identifying G with $G \times \{0\}$ and considering it as a subset of E). Now $L(y) \in G$ and $\theta'(y) \in \operatorname{Im}(L(y))$, hence $\varphi(y)$ is transverse to G. Let $\pi : E \to G$ be the canonical projection. We shall now show that $(\pi \circ \varphi)^* \gamma|_{A \cup W}$ admits an algebraic structure.

Let ν be the normal vector bundle of N in $X \times \mathbb{R}^{l+1}$ with total space

$$E(\nu) = \left\{ (x, v) \in N \times \mathbb{R}^{n+\ell+1} \,\middle|\, x \in N, v \in T_x(X \times \mathbb{R}^{\ell+1}), v \perp T_x N \right\}.$$

By definition of φ, we have $(p \circ \varphi)^* \gamma|_{A \cup W} = \nu|_{A \cup W}$. Therefore, it suffices to prove that $\nu|_A$ and $\nu|_W$ admit an algebraic structure. By (4) there exists a neighborhood U of W in $X \times \mathbb{R}^{\ell+1}$ such that

$$U \cap N = U \cap (W \times \mathbb{R}).$$

Since $W \times \mathbb{R}$ and $X \times \mathbb{R}^{\ell+1}$ are nonsingular affine real algebraic varieties, the normal bundle ν' of $W \times \mathbb{R}$ in $X \times \mathbb{R}^{\ell+1}$ is algebraic (cf. p. 15). Furthermore, $\nu'|_W = \nu|_W$, thus $\nu|_W$ admits an algebraic structure. Concerning $\nu|_A$, observe that by (4)

$$\tau_N|_A = \tau_M|_A \oplus \varepsilon_A^1.$$

Since, by assumption, $\tau_M|_A$ admits an algebraic structure, we obtain that $\tau_N|_A$ also admits an algebraic structure. By definition of ν,

$$\tau_N|_A \oplus \nu|_A = \tau_{X\times\mathbb{R}^{\ell+1}}|_A ,$$

which in view of Proposition 3.1.8 implies that $\nu|_A$ admits an algebraic structure.

Part 3. We prove the following claim.

Claim 2. There exits a smooth mapping $\varphi' : T \to E$ approximating φ arbitrarily close in $C^\infty(T, E)$, with $(\varphi')^{-1}(G) = N$ and $\varphi'|_{A\cup W}$ regular.

Proof of Claim 2. The mapping $\rho\circ\varphi : T \to G$ defined in Part 2 is smooth and $A \cup W$ is an algebraic subset contained in the compact smooth manifold T. Furthermore, we have proved in Part 2 that the vector bundle $(\pi\circ\varphi)^*|_{A\cup W}$ admits an algebraic structure. By Proposition 3.1.9, it follows that there exists a smooth mapping $\psi : T \to G$, arbitrarily close to $\pi\circ\varphi$ in $C^\infty(T, G)$ such that $\psi|_{A\cup W}$ is regular. Define the mapping $\varphi' : T \to E$ by

$$\varphi'(y) = \big(\psi(y), \psi(y)\cdot\theta'(y)\big) \quad \text{for all} \quad y \in T,$$

and set $N' = (\varphi')^{-1}(G)$. Observe that the set $\varphi'(N)$ is contained in G, because $\theta'(y) = 0$ for $y \in N$. If $\psi : T \to G$ is close to the mapping $\pi\circ\varphi : T \to G$, $y \mapsto L(y)$, then φ' is close to the mapping $\varphi : T \to E$, $y \mapsto (L(y), \theta'(y))$. Then taking φ' sufficiently close to φ, it follows from the Thom's isotopy theorem (Thm. 3.2.4) that

$$\varphi' \text{ is transverse to } G,\ N' \cap \partial T = \emptyset, \text{ and } H' \text{ is diffeomorphic to } N.$$

Since $\varphi'(N) \subset G$, we have that $N \subset (\varphi')^{-1}(G) = N'$ and because N is a compact manifold, it follows that $N' = N$.

Part 4. Recall that $(h_X, h_Y) : P \to X \times Y$ is the map defined by

$$(h_X, h_Y)|_M = (i, f) \quad \text{and} \quad (h_X, h_Y)|_W = (s_X, s_Y),$$

where $(s_X, s_Y) : W \to X \times Y$ is a regular mapping, and that $H : N \to Y$ is a smooth extension of $h_Y : P \to Y$. Define the smooth mapping $H' : T \to Y$ by $H' = H \circ \rho$. Clearly,

$$H'\big|_M = f \quad \text{and} \quad H'\big|_W = s_Y.$$

Consider the smooth mapping $F = (H', \varphi') : T \to Y \times E$. We can assume that $Y \times E$ is an algebraic subset of $\mathbb{R}^m$ for some m. Since $A \cup W$ is a quasi-regular subset of T and the mapping

$$F|_{A \cup W} : A \cup W \to Y \times E$$

is regular, it follows from Proposition 3.2.5, that there exist a Zariski closed subset Z of $X \times \mathbb{R}^{\ell+1} \times \mathbb{R}^m$ containing $A \cup W$, a positive real number ε, a smooth mapping $\beta : T \to \mathbb{R}^m$, and a regular mapping $r : Z \to Y \times E$ such that

(6) $\beta|_{A \cup W} = 0$ and β is a close approximation in $C^\infty(T, \mathbb{R}^m)$ of the constant mapping $T \to \mathbb{R}^m$, $t \mapsto 0$.

(7) The set $Z_0 = \{(y, v) \in Z \mid y \in T, \|u\| < \varepsilon\}$ is a compact submanifold (with boundary) of $\operatorname{Reg}(Z)$, $\dim Z_0 = \dim Z$, $(y, \beta(y)) \in Z_0$ for all $y \in T$, and the mapping $\sigma : T \to Z_0$, $\sigma(y) = (y, \beta(y))$, is a C^∞ diffeomorphism.

(8) $r \circ \sigma$ is a close approximation of F in $C^\infty(T, Y \times E)$ and $r|_{A \cup W} = F|_{A \cup W}$.

Note that the mapping $F = (H', \varphi')$ is transverse to the set $Y \times G$ and that $F^{-1}(Y \times G) = N$ with $N \cap \partial T = \emptyset$. Applying Thom's isotopy theorem and using (6)$-$(8), it follows that the mapping $r \circ \sigma : T \to Y \times E$ is transverse to $Y \times G$, and if $N_1 = (r \circ \sigma)^{-1}(Y \times G)$, then $N_1 \cap \partial T = \emptyset$. Furthermore, there exists a smooth embedding $e_1 : N \to T$, close to the inclusion mapping $N \to T$ in $\mathrm{Emb}^{\infty}(N, T)$ such that

$$e_1(N) = N_1, \text{ and } e_1(y) = y \text{ for all } y \in A \cup W.$$

In particular,

$$A \cup W \subset N_1 \text{ and } A \subset M_1 = e_1(M).$$

Part 5. Let $e : N_1 \to N$ be the smooth diffeomorphism determined by

$$e(e_1(y)) = y \text{ for all } y \in N.$$

Define the smooth mappings $f_1 : M \to Y$ and $\delta_1 : N_1 \to \mathbb{R}$ by

$$f_1 = f \circ e|_{M_1} \text{ and } \delta_1 = \delta \circ e.$$

By construction,

$$\begin{aligned} &f|_A = f_1|_A, \ \delta_1 \text{ is transverse to } 0 \text{ in } \mathbb{R}, \text{ and} \\ &\delta_1^{-1}(0) = e^{-1}\left(\delta^{-1}(0)\right) = e^{-1}(M \cup W) = e_1(M) \cup W = M_1 \cup W. \end{aligned}$$

It suffices to show that (M_1, f_1) admits an algebraic approximation in $X \times \mathbb{R}P^k$ relative to A, where $k = n + \ell + 1 + m$.

Let $M_2 = \sigma(M_1)$ and let the mapping $f_2 : M_2 \to Y$ be defined by $f_2 = f_1 \circ (\sigma^{-1}|_{M_2})$ (where $\sigma : T \to Z_0$ is the mapping defined in

(7)). Then it is even sufficient to show that the pair (M_2, f_2) admits an algebraic approximation in $X \times \mathbb{R}P^k$ relative to A. Clearly

$$A \subset M_2 \quad \text{and} \quad f_2|_A = f|_A \,.$$

Set $N_2 = \sigma(N_1)$ and define the smooth function $\delta_2 : N_2 \to \mathbb{R}$ by

$$\delta_2 = \delta_1 \circ \sigma^{-1}\big|_{N_2} \,.$$

Then $M_2 \cup W \subset N_2$, $\delta_2^{-1}(0) = M_2 \cup W$, and δ_2 is transverse to 0 in $\mathbb{R}$. It is clear that W is a Zariski closed subset of $X \times \mathbb{R}P^k$. Observe that, since $N_1 = (r \circ \sigma)^{-1}(Y \times G)$,

$$N_2 = (r|_{Z_0}^{-1})\,(Y \times G)\,,$$

and that N_2 is a union of some connected components of the Zariski closure C' of $r^{-1}(Y \times G)$ in $X \times \mathbb{R}P^k$. Moreover, if C is the Zariski closure of N_2 in C', then each point in N_2 is a nonsingular point of C. From Part 4 we have that $r|_{A\cup W} = F|_{A\cup W} = (H', \varphi')|_{A\cup W}$ and $H'|_M = f$. Hence if $p : Z \to Y$ is the composition of $r : Z \to Y \times E$ and the canonical projection $Y \times E \to Y$, then p is a regular mapping, and

$$p|_A = f_2|_A \,.$$

Now N_2 is a nonsingular affine real algebraic variety, $A \cup W$ is a quasi-regular subset of $X \times \mathbb{R}P^k$, and $\delta_2 : N_2 \to \mathbb{R}$ is a smooth mapping such that $\delta_2|_{A\cup W}$ is regular. Then it follows from Proposition 3.1.5 that there exits a regular function

$$q : X \times \mathbb{R}P^k \to \mathbb{R}$$

such that $q|_{N_2}$ is arbitrarily close to δ_2 in $C^\infty(N_2)$ and

$$q|_{A\cup W} = \delta_2|_{A\cup W} = 0.$$

We may also assume that $q(C\backslash N_2) \subset \mathbb{R}\backslash\{0\}$.

Recall that $\delta_2 : N_2 \to \mathbb{R}$ is a smooth function which is transverse to 0 in $\mathbb{R}$, with $\delta_2^{-1}(0) = M_2 \cup M$ and $\delta_2^{-1}(0) \cap \partial N_2 = \emptyset$. Then by Thom's isotopy theorem, there exists a smooth embedding $j : M_2 \cup W \to N_2$ close to the inclusion mapping $M_2 \cup W \to N_2$ in $\mathrm{Emb}^\infty(M_2 \cup W, N_2)$ such that $j(M_2 \cup W) = (q|_{N_2})^{-1}(0)$ and $j(y) = y$ for all $y \in A \cup W$. Observe that

$$(q|_{N_2})^{-1}(0) = (q|_C)^{-1}(0)$$

is a Zariski closed subset of $X \times \mathbb{R}P^k$ containing $A \cup W$. Moreover, since $q|_{N_2}$ is transverse to 0, we obtain that $(q|_{N_2})^{-1}(0)$ is nonsingular of dimension $\dim W$ by Proposition 3.2.7. Hence, by Proposition 3.2.6, the set

$$V = (q|_{N_2})^{-1}(0)\backslash W$$

is also a Zariski closed nonsingular subset of $X \times \mathbb{R}P^k$. Note that $V = j(M_2)$ and that $g : V \to Y$ defined by $g = p|_W$ is a regular mapping such that $g|_A = f_2|_A$. Then the pair (V, g) is an algebraic approximation of (M_2, f_2) in $X \times \mathbb{R}P^k$ relative to A. This concludes the proof of Theorem 3.2.8. □

4 An uncountable family of nonisomorphic models with prescribed H^1_{alg}

Using the fact that there exist uncountably many nonisogenous complex elliptic curves defined over $\mathbb{R}$, we construct an uncountable family of mutually nonisomorphic algebraic models $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ of a given compact connencted smooth manifold M, such that $H^1_{\mathrm{alg}}(X_\alpha,\ \mathbb{Z}/2)$ is isomorphic to G for each $\alpha \in \mathcal{A}$, where G is a subgroup of $H^1(M, \mathbb{Z}/2)$ containing $w_1(M)$.

4.1 An uncountable family of nonisomorphic real algebraic varieties

Recall that a *complex elliptic curve defined over* $\mathbb{R}$ is an abelian variety of dimension 1 defined over $\mathbb{R}$. Each such variety is isomorphic to a nonsingular complex projective cubic defined over $\mathbb{R}$, equipped with an appropriate group operation.

Lemma 4.1.1 *There exist an uncountable family of complex elliptic curves defined over* $\mathbb{R}$ *which are mutually nonisogenous.*

Proof. Since two elliptic curves defined over $\mathbb{C}$ are isomorphic if and only if they have the same j-invariant, and since for every $c \in \mathbb{C}$ there exists an elliptic curve E defined over $\mathbb{C}$ with $j(E) = c$

[35, Prop. III.1.4], there are uncountably many isomorphy classes of elliptic curves defined over $\mathbb{C}$. Each elliptic curve defined over $\mathbb{C}$ corresponds to a complex torus $\mathbb{C}/\Lambda_\tau$, where $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$ and Λ_τ is a lattice $\mathbb{Z}\tau + \mathbb{Z}$ in $\mathbb{C}$ [35, Cor. 5.1.1]. It can be proved that two complex tori $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ are isogenous if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) = \left\{ \begin{pmatrix} r & s \\ t & v \end{pmatrix} \middle| \; r, s, t, v \in \mathbb{Q},\ rv - st > 0 \right\}$$

such that $\tau' = \frac{a\tau+b}{c\tau+d}$ [23, Thm. 11.1.4]. This implies that each isogeny class contains only countably many curves up to isomorphism. Choosing one curve from each isogeny class, nonisomorphic with the others, we obtain an uncountable family of mutually nonisogenous elliptic curves defined over $\mathbb{C}$. Since a complex elliptic curve E is real if and only if $j(E)$ is in $\mathbb{R}$, and, up to isomorphism over $\mathbb{R}$, there are exactly 2 complex elliptic curves defined over $\mathbb{R}$ with the same j-invariant, the theorem follows. $\square$

Using Lemma 4.1.1 and the properties of Albanese varieties as discussed in Section 1.5, one can distinguish between real algebraic varieties. The following result will make this clear, (cf. [6]).

Theorem 4.1.2 *Let $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be an uncountable family of complex elliptic curves in $\mathbb{C}P^2$ defined over $\mathbb{R}$ such that E_α is not isogenous to E_β for $\alpha \neq \beta$. For each α, let C_α be the real part of E_α and let $\{f_\alpha : X_\alpha \to C_\alpha\}_{\alpha\in\mathcal{A}}$ be an uncountable family of nonconstant regular mappings, where each X_α is an affine nonsingular real algebraic variety. Then there exists an uncountable subset $\mathcal{B}$ of $\mathcal{A}$ such that the varieties*

$\{X_\alpha\}_{\alpha\in\mathcal{B}}$ *are mutually not birationally equivalent.*

Proof. Let $\alpha \in \mathcal{A}$ and consider the subset

$$\mathcal{A}_\alpha = \{\beta \in \mathcal{A} \mid X_\beta \text{ is birationally equivalent to } X_\alpha\}.$$

It is sufficient to show that $\mathcal{A}_\alpha$ is finite for each $\alpha \in \mathcal{A}$.

Taking the nonsingular projective complexification $X'_{\alpha\mathbb{C}}$ of X_α in $\mathbb{C}P^n$, we can extend the mapping $f_\alpha : X_\alpha \to C_\alpha$ to a nonconstant rational mapping $X'_{\alpha\mathbb{C}} \to E_\alpha$. By the universal property of Albanese varieties, the rational mapping $X'_{\alpha\mathbb{C}} \to E_\alpha$ induces a nonzero, and thus surjective, homomorphism

$$\tilde{f}_\alpha : Alb\,(X_\alpha) \to E_\alpha.$$

Now $\mathrm{Alb}\,(X_\alpha)$ is isogenous to the product $Y_1^{n_1} \times \ldots \times Y_k^{n_k}$ of simple abelian varieties. By Poincaré's reducibility theorem (p. 17), $\mathrm{Alb}(X_\alpha)$ and $\ker \tilde{f}_\alpha \times E_\alpha$ are isogenous. Therefore, E_α must be isogeneous to one of the simple abelian varieties $Y_1, \ldots, Y_k$.

If $\mathrm{Alb}\,(X_\beta)$ is isomorphic to $\mathrm{Alb}\,(X_\alpha)$, where $\alpha \neq \beta$, then one of the factors Y_i is isogenous to E_β. Since E_α and E_β are not isogenous for $\alpha \neq \beta$, it follows that the set

$$\mathcal{A}'_\alpha = \{\beta \in \mathcal{A} \mid \mathrm{Alb}\,(X_\alpha) \text{ is isomorphic to } \mathrm{Alb}\,(X_\beta)\}$$

has at most k elements. Since $\mathcal{A}_\alpha \subset \mathcal{A}'_\alpha$, the set $\mathcal{A}_\alpha$ is finite. □

4.2 Varieties of type $\mathcal{A}$

Consider a complex algebraic variety X defined over $\mathbb{R}$. Then we can also regard X as a variety defined over $\mathbb{C}$. We denote this variety by

$X_{\mathbb{C}}$. Similarly, a complex algebraic vector bundle $\xi = (E, \pi, X)$ defined over $\mathbb{R}$ is, in particular, a complex algebraic vector bundle, which we denote by $\xi_{\mathbb{C}} = (E_{\mathbb{C}}, \pi_{\mathbb{C}}, X_{\mathbb{C}})$.

Definition 4.2.1 *Let X be a nonsingular complex projective algebraic variety defined over $\mathbb{R}$. Then the* Picard group $\mathrm{Pic}(X)$ *of X is the group of isomorphism classes (over $\mathbb{R}$) of complex algebraic line bundles defined over $\mathbb{R}$.*

Let X be a complex algebraic variety defined over $\mathbb{R}$. Then there is a canonical involution $\sigma_X : X \to X$ defined as follows.

Let $x \in X$ and let U be an affine open subset of X containing x. Let $\varphi : U \to V$ be a regular isomorphism defined over $\mathbb{R}$, where V is a closed subvariety of $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C})$. We set

$$\sigma_X(x) = \varphi^{-1}(\tau_n(\varphi(x))),$$

where $\tau_n : \mathbb{C}^n \to \mathbb{C}^n$ is the involution determined by the complex conjugation,

$$\tau_n(\lambda_1, \ldots, \lambda_n) = \left(\bar{\lambda}_1, \ldots, \bar{\lambda}_n\right).$$

This definition does not depend on the choice of U, V, and φ. Furthermore, σ_X is an involution. Since X and $X_{\mathbb{C}}$ have the same underlying set, we can regard σ_X as an involution on $X_{\mathbb{C}}$.

Let $\xi = (E, \pi, X)$ be a complex algebraic vector bundle defined over $\mathbb{R}$ and let $\Gamma(\xi)$ be the space of all global algebraic sections of ξ. Then there exist a canonical involution $\sigma_\xi : \Gamma(\xi) \to \Gamma(\xi)$ defined by

$$\sigma_\xi(s) = \sigma_E \circ s \circ \sigma_X \quad \text{for } s \in \Gamma(\xi).$$

Note that σ_ξ is a homomorphism of additive groups and $\sigma_\xi(\lambda s) = \bar{\lambda}\sigma_\xi(s)$ for $\lambda \in \mathbb{C}$ and $s \in \Gamma(\xi)$. Now we can prove the following result [10].

Theorem 4.2.2 *Let X be a complex nonsingular projective algebraic variety defined over $\mathbb{R}$. If $X_\mathbb{C}$ is irreducible, then the homomorphism*

$$\operatorname{Pic}(X) \to \operatorname{Pic}(X_\mathbb{C}), \quad [\xi] \to [\xi_\mathbb{C}]$$

is injective.

Proof. Let $\xi = (E, \pi, X)$ be a complex algebraic line bundle on X defined over $\mathbb{R}$ (Def. 1.4.12). We have to show that if the line bundle $\xi_\mathbb{C} \in \operatorname{Pic}(X_\mathbb{C})$ is trivial, then ξ is trivial as well.

Assume $\xi_\mathbb{C}$ is trivial and let $u : X_\mathbb{C} \to E_\mathbb{C}$ be a global algebraic section defined over $\mathbb{C}$, that is, $u(x) \neq 0$ for all $x \in X_\mathbb{C}$. Since $X_\mathbb{C} \subset \mathbb{P}^n_\mathbb{C}(\mathbb{C})$ is compact, the modulus of a global holomorphic function on $X_\mathbb{C}$ must take a maximum on $X_\mathbb{C}$. By the maximum modulus theorem, such a function is constant, hence $\mathcal{O}_{X_\mathbb{C}}(X_\mathbb{C}) = \Gamma(\xi_\mathbb{C}) = \mathbb{C}$. It follows that the mapping

$$\varphi : \mathbb{C} \to \Gamma(\xi_\mathbb{C}), \quad \lambda \mapsto \lambda u$$

is a $\mathbb{C}$-linear isomorphism. Define an involution $\tau : \mathbb{C} \to \mathbb{C}$ by

$$\tau = \varphi^{-1} \circ \sigma_\xi \circ \varphi.$$

Setting $\mu = \tau(1)$, we get

$$\mu u = \varphi(\mu) = \varphi\left(\left(\varphi^{-1} \circ \sigma_\xi \circ \varphi\right)(1)\right) = \sigma_\xi(u).$$

Thus for $\lambda \in \mathbb{C}$,

$$\begin{aligned}\tau(\lambda) &= \left(\varphi^{-1} \circ \sigma_\xi \circ \varphi\right)(\lambda) = \varphi^{-1}\left(\sigma_\xi(\lambda u)\right)\varphi^{-1}\left(\bar{\lambda}\sigma_\xi(u)\right) \\ &= \varphi^{-1}\left(\bar{\lambda}\mu u\right) = \bar{\lambda}\mu.\end{aligned}$$

Since τ is an involution,

$$\lambda = (\tau \circ \tau)(\lambda) = \tau\left(\bar{\lambda}\mu\right) = \overline{\left(\bar{\lambda}\mu\right)}\mu = \lambda\bar{\mu}\mu,$$

and hence $\bar{\mu}\mu = 1$. This implies that for the complex numbers $\mu = a + bi$ and $z = b + (1-a)\,i$, $(a, b \in \mathbb{R})$, we have $\tau(z) = \bar{z}\mu = z$. Note that the section $v = \varphi(z) = zu \in \Gamma(\xi_{\mathbb{C}})$ satisfies

$$\sigma_\xi(v) = \sigma_\xi(\varphi(z)) = \varphi(\tau(z)) = \varphi(z) = v$$

and $v(x) = zu(x) \neq 0$ for all $x \in X_{\mathbb{C}}$. Since $v = \sigma_\xi(v) = \sigma_E \circ v \circ \sigma_X$, it follows that $v = s_{\mathbb{C}}$ for some algebraic section $s : X \to E$ defined over $\mathbb{R}$. Therefore, ξ is trivial. □

The Picard group $\operatorname{Pic}(X)$ can thus be considered as a subgroup of the group $\operatorname{Pic}(X_{\mathbb{C}})$ of isomorphy classes of all complex algebraic line bundles over $X_{\mathbb{C}}$. This brings us to the following definition.

Definition 4.2.3 *Let X be a complex nonsingular irreducible projective algebraic variety defined over $\mathbb{R}$. Then X is said to be of* type $\mathcal{A}$ *if*

$$\operatorname{Pic}(X) = \operatorname{Pic}(X_{\mathbb{C}}).$$

An affine real algebraic variety V is said to be of type $\mathcal{A}$ *if there exists a complex projective algebraic variety X defined over $\mathbb{R}$, of type $\mathcal{A}$, such that V is biregularly isomorphic to the real part of X.*

Example 4.2.4 *Consider the variety $\mathbb{R}P^n$ as the real part of $\mathbb{C}P^n$. The variety $\mathbb{C}P^n$, and therefore $\mathbb{R}P^n$, is a variety of type $\mathcal{A}$. More generally, the variety $\mathbb{R}P^{n_1} \times \ldots \times \mathbb{R}P^{n_k}$ is of type $\mathcal{A}$.*

Not every algebraic variety is of type $\mathcal{A}$. For example, no complex projective nonsingular curve defined over $\mathbb{R}$ of genus $g \geq 1$ is of type $\mathcal{A}$.

Next we give a result describing a method of constructing varieties of type $\mathcal{A}$ (cf. [5, p. 594]). We need the following definition.

Definition 4.2.5 *Let X be a compact affine nonsingular real algebraic variety of dimension d. A smooth submanifold M of X is said to be* admissible *if $M = H_1 \cap \ldots \cap H_c$, $c = \operatorname{codim} M$, where $H_1, \ldots, H_c$ are compact smooth hypersurfaces of X which are in general position at each point of M and the homology classes in $H_{d-1}(X, \mathbb{Z}/2)$ represented by the H_i are in $H_{d-1}^{\text{alg}}(X, \mathbb{Z}/2)$, $i = 1, \ldots, c$.*

Theorem 4.2.6 *Let X be a compact affine nonsingular irreducible real algebraic variety of type $\mathcal{A}$, and let M be an admissible smooth submanifold of X with $\dim M \geq 2$. Then, given a neighborhood $\mathcal{U}$ of the identity mapping in the set $\operatorname{Diff}(X)$ (the set of all diffeomorphisms of X equipped with the C^∞ topology), one can find a diffeomorphism $\varphi \in \mathcal{U}$ such that $Y = \varphi(M)$ is an irreducible nonsingular real algebraic variety of type $\mathcal{A}$ and the restriction homomorphism*

$$H^1_{\text{alg}}(X, \mathbb{Z}/2) \to H^1_{\text{alg}}(Y, \mathbb{Z}/2)$$

is surjective.

4.3 An uncountable family of nonisomorphic algebraic models

Let M be a compact connected smooth 3-dimensional manifold and let G be a subgroup of $H^1(M, \mathbb{Z}/2)$. Suppose that there exists an affine nonsingular real algebraic variety X and a diffeomorphism $\varphi : X \to M$ such that $\varphi^*(G) = H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. For any algebraic vector bundle over X, we have ([3, p. 304])

Theorem 4.3.1 *If ξ is an algebraic vector bundle over a compact nonsingular affine real algebraic variety X, then the total Stiefel-Whitney class $w(\xi)$ of ξ belongs to the ring $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$.*

Since the tangent bundle τ_X is algebraic, Theorem 4.3.1 implies that the first Stiefel-Whitney class $w_1(X)$ is contained in $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$, and therefore, that $w_1(M) \in G$. A necessary condition for the existence of an algebraic model X of M and a diffeomorphism $\varphi : X \to M$ with $\varphi^*(G) = H^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$, is that the first Stiefel-Whitney class $w_1(M)$ of M is contained in G. This turns out to be the only obstruction in proving the existence of such algebraic models, at least if $\dim M \geq 3$. With the next result we see that an uncountable family of such models exists.

Theorem 4.3.2 *Let M be a compact connected smooth manifold of dimension $m \geq 3$ and let G be a subgroup of $H^1(M, \mathbb{Z}/2)$ containing the first Stiefel-Whitney class $w_1(M)$ of M. Then there exists an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraic models of M and a family of smooth diffeomorphisms $\{h_\alpha : X_\alpha \to M\}_{\alpha \in \mathcal{A}}$ such that* (*i*) X_α and X_β are not birationally equivalent for $\alpha \neq \beta$,

(*ii*) $h_\alpha^*(G) = H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2)$ for each $\alpha \in \mathcal{A}$.

Proof. The proof consists of six parts.

Part 1. We may assume that M is a smooth submanifold of

$$P = \mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_\ell},$$

where $n_1, \ldots, n_\ell$ are odd integers (in particular, P is orientable), and that

$$r\left(H^1(P, \mathbb{Z}/2)\right) = G, \tag{4.1}$$

where $r : H^1(P, \mathbb{Z}/2) \to H^1(M, \mathbb{Z}/2)$ is the restriction homomorphism. Indeed, let $\{w_1, \ldots w_\ell\}$ be the set of generators of G and let $\{a_1, \ldots, a_\ell\}$ be the set of generators of groups $H^1(\mathbb{R}P^{n_j}, \mathbb{Z}/2)$, where $j = 1, \ldots, \ell$ and n_j is odd with $n_j \geq 2m+1$. Pick ℓ smooth mappings $\varphi_j : M \to \mathbb{R}P^{n_j}$ such that $w_j = \varphi_j^*(a_j)$. Since M is compact, the mapping $(\varphi_1, \ldots, \varphi_\ell)$ can be approximated in the C^∞ topology by a smooth embedding

$$\psi : M \to \mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_\ell}$$

(cf. Thm. 3.1.1). Then $\psi_j^*(a_j) = w_j$ and by replacing, if necessary, M by $\psi(M)$, we can consider M being contained in P. Then we have the inclusion mapping $i : M \to P$ and thus the restriction homomorphism r, and obtain $r\left(H^1(P, \mathbb{Z}/2)\right) = G$.

Part 2. Let $H^1(M, \mathbb{Z}/2) \backslash G = \{v_1, \ldots, v_k\}$. By Theorem 2.2.4 there exists an element $u_i \in H^{m-1}(M, \mathbb{Z}/2)$ for each $i = 1, \ldots, k$ such that

(i) $v_i \cup u_i \neq 0$ and $w \cup u_i = 0$ for all $w \in G$,

(ii) $D_M(u_i) = [C_i]_M$ in $H_1(M, \mathbb{Z}/2)$ for some compact connected smooth curve C_i in M, embedded with trivial normal vector bundle.

Moreover, since $\dim M \geq 3$, we may assume that $C_i \cap C_j = \emptyset$ for $i \neq j$. The curves C_i are also embedded in P with trivial normal vector bundle. Indeed, since $\nu_{C_i,P} \oplus \tau_{C_i} = \tau_P|_{C_i}$, the normal bundle $\nu_{C_i,P}$ is orientable. Furthermore, $\nu_{C_i,P} = \xi \oplus \varepsilon_{C_i}$, where ξ is some line bundle over C_i and ε_{C_i} is the trivial bundle. Then it follows that ξ is orientable and thus trivial.

Since the C_i have trivial normal vector bundles in M and P, one can construct k tori T_i bounded in M by revolving a small circle along each curve C_i such that the tori don't intersect with each other. Furthermore, one can construct these tori in such a way that they form the boundary of a compact smooth submanifold W of M with $\dim W = 3$, and such that the normal vector bundles of W in P and of $Y = \partial W$ in M are trivial.

Part 3. Claim. If $\rho : H^1(M, \mathbb{Z}/2) \to H^1(Y, \mathbb{Z}/2)$ is the restriction homomorphism, then

$$\rho(G) = 0, \tag{4.2}$$

$$\rho(v_i) \neq 0 \quad \text{for} \quad i = 1, \ldots, k. \tag{4.3}$$

Proof of (4.2). Let $w \in G$. Then $w \cup u_i = 0$ for $i = 1, \ldots, k$. Arguing as in Theorem 2.2.4, part (ii), one can find a smooth hypersurface H of M with $[H]_M$ Poincaré dual to w, such that $C_i \cap H = \emptyset$ for all i. We can choose H in such a way that $H \cap Y = \emptyset$. Since ρ is the restriction homomorphism, it follows that $\rho(w) = 0$.

Proof of (4.3). Let $f_i : C_i \to M$ be an inclusion mapping for each i. Since $C_i \subset Y$, it suffices to prove that $f_i^*(v_i) \neq 0$ for all $i = 1, \ldots, k$.

Let $\tilde{v}_i \in H_{m-1}(M, \mathbb{Z}/2)$ be Poincaré dual to v_i. Again arguing as in the proof of Theorem 2.2.4 (ii), one can find a compact smooth hypersurface N_i in M, transverse to C_i, such that $\tilde{v}_i = [N_i]_M$. By Corollary 2.1.5, we may assume that f_i is transverse to N_i. Then it follows from Theorem 2.1.6 that

$$(D_{C_i} \circ f_i^* \circ D_M^{-1})([N_i]_M) = [f_i^{-1}(N_i)]_{C_i} = [C_i \cap N_i]_{C_i}.$$

Therefore, $f_i^*(v_i)$ is Poincaré dual to $[C_i \cap N_i]_{C_i}$. By Corollary 2.1.7,

$$v_i \cup u_i = D_M^{-1}([N_i]_M) \cup D_M^{-1}([C_i]_M) = D_M^{-1}([C_i \cap N_i]_M) \neq 0.$$

Hence $[C_i \cap N_i]_M \neq 0$, which implies that the number of points in $C_i \cap N_i$ is odd. It follows that $[C_i \cap N_i]_{C_i} \neq 0$, and thus $f_i^*(v_i) \neq 0$.

Part 4. The closed submanifold Y of P is the boundary of a closed submanifold W of P embedded with a trivial normal vector bundle. By Proposition 1.4.7, this implies that there exists a smooth mapping $F : P \to \mathbb{R}^c$, where $c = \operatorname{codim} Y = \dim P - 2$, such that $0 \in \mathbb{R}^c$ is a regular value of F and $F^{-1}(0) = Y$. Let $F = (F_1, \ldots, F_c)$, then

$$Y = F_1^{-1}(0) \cap \ldots \cap F_c^{-1}(0),$$

where $F_1^{-1}(0), \ldots, F_c^{-1}(0)$ are compact smooth hypersurfaces of P (after a small perturbation, if necessary) which are in general position at each point of Y. Thus Y is an admissible smooth submanifold of P (Def. 4.2.5).

Let U be a neighborhood of the identity mapping in Diff (P). By Example 4.2.4, P is a variety of type $\mathcal{A}$. Then by Theorem 4.2.6 there exists a diffeomorphism $g : P \to P$ in U such that $g(Y)$ is an irreducible nonsingular real algebraic variety of type $\mathcal{A}$ and the restriction homomorphism

$$H^1_{\text{alg}}(P, \mathbb{Z}/2) \to H^1_{\text{alg}}(g(Y), \mathbb{Z}/2)$$

is surjective. Since P is totally algebraic (Example 2.3.2), it follows that $H^1_{\text{alg}}(g(Y), \mathbb{Z}/2)$ is equal to the image of the restriction homomorphism

$$H^1(P, \mathbb{Z}/2) \to H^1(g(Y), \mathbb{Z}/2).$$

Replacing M and Y by $g(M)$ and $g(Y)$, respectively, we may assume *that Y is a nonsingular algebraic subvariety of P*, $Y \subset M$, and

$$(\rho \circ r)\left(H^1(P, \mathbb{Z}/2)\right) = H^1_{\text{alg}}(Y, \mathbb{Z}/2). \tag{4.4}$$

It follows from (4.1), (4.2), and (4.4) that

$$H^1_{\text{alg}}(Y, \mathbb{Z}/2) = 0. \tag{4.5}$$

Part 5. Consider the triple $Y \subset M \subset P$ constructed above. Let $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ be an uncountable family of complex elliptic curves in $\mathbb{C}P^2$ defined over $\mathbb{R}$ such that E_α is not isogenous to E_β for $\alpha \neq \beta$ and let C_α be the real part of E_α. For each $\alpha \in \mathcal{A}$, choose a nonconstant smooth mapping $f_\alpha : M \to C_\alpha$ which is constant on Y. Then the restriction $f_\alpha|_Y$ is regular. Since Y has a trivial normal bundle $\nu_{Y,M}$ in M and the tangent bundle τ_Y is algebraic, the tangent bundle $\tau_M|_Y = \tau_Y \oplus \nu_{Y,M}$ admits an algebraic structure. Furthermore, Theorem 2.4.3, 2.4.4, and Example 2.4.5 imply that $\mathfrak{N}_*^{\mathrm{alg}}(P \times C_\alpha) = \mathfrak{N}_*(P \times C_\alpha)$, so that the bordism class of $(M, (i, f_\alpha))$ is algebraic. Then it follows from Theorem 3.2.8 that there exists a nonnegative integer n such that (M, f_α) admits an algebraic approximation in $P \times \mathbb{R}^n$ relative to Y. That is, there exists a smooth embedding $e_\alpha : M \times \{0\} \to P \times \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \times \{0\} \to P \times \mathbb{R}^n$, a Zariski closed nonsingular subvariety X_α of $P \times \mathbb{R}^n$, and a mapping $g_\alpha : X_\alpha \to C_\alpha$ such that

$$X_\alpha = e_\alpha(M \times \{0\}), \quad e_\alpha(x) = (x, 0) \text{ for all } x \in Y, \text{ and}$$
$$g_\alpha \text{ is a regular mapping with } g_\alpha \circ e_\alpha \text{ close to } f_\alpha \text{ in the } C^\infty \text{ topology.}$$

Clearly, g_α is not constant if $g_\alpha \circ e_\alpha$ is sufficiently close to f_α. By Theorem 4.1.2, there exists an uncountable subset $\mathcal{B}$ of $\mathcal{A}$ such that $\{X_\alpha\}_{\alpha \in \mathcal{B}}$ is an uncountable family of mutually not birationally equivalent algebraic models of the manifold M.

Part 6. Equation (4.1) implies that there exists a diffeomorphism $h_\alpha : X_\alpha \to M$ such that $h_\alpha^*(G)$ is equal to the image of the restriction homomorphism

$$j_\alpha^* : H^1(P \times \mathbb{R}^n, \mathbb{Z}/2) \to H^1(X_\alpha, \mathbb{Z}/2).$$

In particular, since $H^1(P, \mathbb{Z}/2) = H^1_{\mathrm{alg}}(P, \mathbb{Z}/2)$, we have

$$h_\alpha^*(G) \subset H^1_{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2).$$

Let $\sigma : H^1(X_\alpha, \mathbb{Z}/2) \to H^1(Y, \mathbb{Z}/2)$ be the homomorphism induced by the mapping $Y \to X_\alpha$ defined by $y \mapsto (y, 0)$. Since the mapping

$$h_\alpha^* : H^1(M, \mathbb{Z}/2) \to H^1(X_\alpha, \mathbb{Z}/2)$$

is an isomorphism, relation (4.3), implies that $\sigma(h_\alpha^*(v_i)) \neq 0$. In particular, since by (4.5), $H^1_{\text{alg}}(Y, \mathbb{Z}/2) = 0$, the element $\sigma(h_\alpha^*(v_i))$ is not in $H^1_{\text{alg}}(Y, \mathbb{Z}/2)$ for $i = 1, \ldots, k$. This implies that $h_\alpha^*(v_i)$ is not in $H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2)$ for $i = 1, \ldots, k$. Therefore, $H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2) \subset h_\alpha^*(G)$, and thus $h_\alpha^*(G) = H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2)$ for each $\alpha \in \mathcal{B}$.

This proves the existence of an uncountable family of mutually nonisomorphic algebraic models $\{X_\alpha\}_{\alpha \in \mathcal{B}}$ of the manifold M, together with a family $\{h_\alpha : X_\alpha \to M\}_{\alpha \in \mathcal{A}}$ of smooth diffeomorphism such that $h_\alpha^*(G) = H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2)$. □

4.4 Surfaces

For smooth compact connected surfaces a result similar to Theorem 4.3.2 can be proved.

Let X be a compact nonsingular affine real algebraic surface and define

$$\beta(X) = \dim_{\mathbb{Z}/2} H^1_{\text{alg}}(X, \mathbb{Z}/2)$$
$$\delta(X) = \dim_{\mathbb{Z}/2} \left\{ v \in H^1_{\text{alg}}(X, \mathbb{Z}/2) \mid v \cup v = 0 \right\}.$$

If the surface X is connected and orientable, then $\beta(X) = \delta(X)$. If X is nonorientable and of odd topological genus, then $\beta(X) = \delta(X) + 1$. For connected nonorientable surfaces X of even genus one can have either $\beta(X) = \delta(X)$ of $\beta(X) = \delta(X) + 1$, depending on a chosen model (see [29, Section 49]).

The following result is given in [5, Thm. 1.4].

Theorem 4.4.1 (*i*) *Let M be a compact connected smooth surface of genus g and let k be an integer satisfying*

$$\begin{aligned} 0 \leq k \leq 2g \qquad & \textit{for } M \textit{ orientable,} \\ 1 \leq k \leq g \qquad & \textit{for } M \textit{ nonorientable.} \end{aligned}$$

Then there exists an algebraic model X of M such that $\beta(X) = k$.
(*ii*) *Let M be a compact connected nonorientable smooth surface of even genus g and let k and m be integers satisfying the following properties:*

$$\begin{aligned} k = m + 1, &\quad 2 \leq k \leq g, \\ \textit{or} \quad k = m, &\quad 1 \leq k \leq g - 1. \end{aligned}$$

Then there exists an algebraic model X of M with $\beta(X) = k$ and $\delta(X) = m$.

In order to prove that there are uncountably many algebraic models as in Theorem 4.3.2, we need the following proposition and corollary (compare [5, Thm. 5.5])

Proposition 4.4.2 *Let X_1 and X_2 be compact connected affine non-singular real algebraic surfaces. Let b_i (resp. β_i) be the dimension of the $\mathbb{Z}/2$-vector space $H^1(X_i, \mathbb{Z}/2)$ (resp. $H^1_{alg}(X_i, \mathbb{Z}/2)$), $i = 1, 2$. Then* *there exists an uncountable family* $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ *of mutually nonisomorphic algebraic models of the connected sum* $X_1 \# X_2$ *such that*

$$2^{\beta_1+\beta_2} \leq \textit{order } H^1_{alg}(X_\alpha, \mathbb{Z}/2) \leq 2^{\beta_1+\beta_2} + (2^{b_1} - 2^{\beta_1})(2^{b_2} - 2^{\beta_2}).$$

Proof. Let $\{x_1\} \subset X_{11} \subset X_1$, and $\{x_2\} \subset X_{21} \subset X_2$, where X_{i1} is a connected nonsingular algebraic curve contained in X_i and x_i is a point in X_{i1}, $i = 1, 2$. Let $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be an uncountable family of complex elliptic curves in $\mathbb{C}P^2$ defined over $\mathbb{R}$ such that E_α is not isogenous to E_β for $\alpha \neq \beta$ and let C_α be the real part of E_α. Set $A_\alpha = X_1 \times X_2 \times C_\alpha$. Let

$$\begin{aligned} C_{\alpha 0} &= (X_1 \times X_2 \times \{y_{\alpha 1}\}) \cup (X_1 \times X_2 \times \{y_{\alpha 2}\}), \\ C_{\alpha 1} &= (X_1 \times X_{21} \times \{y_{\alpha 1}\}) \cup (X_{11} \times X_2 \times \{y_{\alpha 2}\}), \\ C_{\alpha 2} &= (X_1 \times \{x_2\} \times \{y_{\alpha 1}\}) \cup (\{x_1\} \times X_2 \times \{y_{\alpha 2}\}), \end{aligned}$$

where $y_{\alpha 1}$, and $y_{\alpha 2}$ are two distinct points in C_α.

Let $B_{\alpha 0}$ be a smooth hypersurface of A_α containing $C_{\alpha 1}$ and diffeomorphic to the connected sum of two connected components of $C_{\alpha 0}$. The hypersurface $B_{\alpha 0}$ can be chosen such that $[B_{\alpha 0}]_{A_\alpha} = [C_{\alpha 0}]_{A_\alpha}$, and since $C_{\alpha 0}$ is an algebraic subset of A_α, we have that $[B_{\alpha 0}]_{A_\alpha}$ is an element of $H_4^{\mathrm{alg}}(A_\alpha, \mathbb{Z}/2)$ ($\dim A_\alpha = 5$). It follows from Corollary 3.2.2 and Remark 3.2.3 that there exists a nonsingular algebraic hypersurface $A_{\alpha 0}$ of A_α, containing $C_{\alpha 1}$, which is diffeotopic to $B_{\alpha 0}$.

Now let $B_{\alpha 1}$ be a smooth 3-dimensional manifold in $A_{\alpha 0}$ containing $C_{\alpha 2}$ and diffeomorphic to the connected sum of two connected components of $C_{\alpha 1}$. Again by Corollary 3.2.2 and Remark 3.2.3, it follows that there exists a nonsingular algebraic 3-fold $A_{\alpha 1}$ in $A_{\alpha 0}$, containing $C_{\alpha 2}$

Diagram for Proposition 4.4.2

$A_\alpha = X_1 \times X_2 \times C_\alpha$ dim 5

$\subset$ $\subset$

$C_{\alpha 1} \subset A_{\alpha 0} \xleftarrow{\text{diffeotopic}} B_{\alpha 0} \supset C_{\alpha 1} = (X_1 \times X_{21} \times \{y_{\alpha 1}\}) \cup (X_{11} \times X_2 \times \{y_{\alpha 2}\})$ dim 4

$B_{\alpha 0} \xrightarrow{\approx} C_{\alpha 01} \# C_{\alpha 02}$

$C_{\alpha 0i}$ are connected components of
$C_{\alpha 0} = (X_1 \times X_2 \times \{y_{\alpha 1}\}) \cup (X_1 \times X_2 \times \{y_{\alpha 2}\})$

$\subset$ $\subset$

$C_{\alpha 2} \subset A_{\alpha 1} \xleftarrow{\text{diffeotopic}} B_{\alpha 1} \supset C_{\alpha 2} = (X_1 \times \{x_2\} \times \{y_{\alpha 1}\}) \cup (\{x_1\} \times X_2 \times \{y_{\alpha 2}\})$ dim 3

$B_{\alpha 1} \xrightarrow{\approx} C_{\alpha 11} \# C_{\alpha 12}$

$C_{\alpha 1i}$ are connected components of
$C_{\alpha 1} = (X_1 \times X_{21} \times \{y_{\alpha 1}\}) \cup (X_{11} \times X_2 \times \{y_{\alpha 2}\})$

$\subset$

$B_{\alpha 2} \supset Y^{\alpha}_{1r},\ Y^{\alpha}_{2s},\ N^{\alpha}_{1t},\ N^{\alpha}_{2u} \subset C_{\alpha 2} = (X_1 \times \{x_2\} \times \{y_{\alpha 1}\}) \cup (\{x_1\} \times X_2 \times \{y_{\alpha 2}\})$ dim 2

$B_{\alpha 2} \xrightarrow{\approx} C_{\alpha 21} \# C_{\alpha 22}$

$C_{\alpha 2i}$ are connected components of
$C_{\alpha 2} = (X_1 \times \{x_2\} \times \{y_{\alpha 1}\}) \cup (\{x_1\} \times X_2 \times \{y_{\alpha 2}\})$

and diffeotopic in $A_{\alpha 0}$ to $B_{\alpha 1}$.

Let $Y_{i1}, \ldots, Y_{ik_i}$, $k_i = 2^{\beta_i}$, be nonsingular algebraic curves in X_i representing all homology classes in $H_1^{\text{alg}}(X_i, \mathbb{Z}/2)$, and let $N_{i1}, \ldots, N_{il_i}$, $l_i = 2^{b_i} - 2^{\beta_i}$, be smooth compact curves in X_i representing all homology classes in $H_1(X_i, \mathbb{Z}/2) \setminus H_1^{\text{alg}}(X_i, \mathbb{Z}/2)$, $i = 1, 2$. We can assume that the Y_{ij} are in general position. Put

$$\begin{aligned}
Y_{1r}^{\alpha} &= Y_{1r} \times \{x_2\} \times \{y_{\alpha 1}\} && \textit{for } r = 1, \ldots, k_1,\\
Y_{2s}^{\alpha} &= \{x_1\} \times Y_{2s} \times \{y_{\alpha 2}\} && \textit{for } s = 1, \ldots, k_2,\\
N_{1t}^{\alpha} &= N_{1t} \times \{x_2\} \times \{y_{\alpha 1}\} && \textit{for } t = 1, \ldots, l_1,\\
N_{2u}^{\alpha} &= \{x_1\} \times N_{2u} \times \{y_{\alpha 2}\} && \textit{for } u = 1, \ldots, l_2,
\end{aligned}$$

Since

$$C_{\alpha 2} = (X_1 \times \{x_2\} \times \{y_{\alpha 1}\}) \cup (\{x_1\} \times X_2 \times \{y_{\alpha 2}\}),$$

it follows that the homology classes $[Y_{1r}^{\alpha}]_{C_{\alpha 2}}$ and $[Y_{2s}^{\alpha}]_{C_{\alpha 2}}$ are in $H_1^{\text{alg}}(C_{\alpha 2}, \mathbb{Z}/2)$. Furthermore, the classes $[N_{1t}^{\alpha}]_{C_{\alpha 2}}$ and $[N_{2u}^{\alpha}]_{C_{\alpha 2}}$ are in $H_1(C_{\alpha 2}, \mathbb{Z}/2) \setminus H_1^{\text{alg}}(C_{\alpha 2}, \mathbb{Z}/2)$. Now choose a smooth surface $B_{\alpha 2}$ in $A_{\alpha 1}$ which is diffeomorphic to the connected sum of two connected components of $C_{\alpha 2}$, and which contains all the Y_{1r}^{α}, Y_{2s}^{α}, N_{1t}^{α}, and N_{2u}^{α}. We can assume that $[B_{\alpha 2}]_{A_{\alpha 1}} = [C_{\alpha 2}]_{A_{\alpha 1}}$. In particular, we have that $[B_{\alpha 2}]_{A_{\alpha 1}} \in H_2^{\text{alg}}(A_{\alpha 1}, \mathbb{Z}/2)$. Applying Corollary 3.2.2, one can choose a diffeomorphism $h_{\alpha} : A_{\alpha 1} \to A_{\alpha 1}$ arbitrarily close to the identity mapping of $A_{\alpha 1}$ such that $X_{\alpha} = h_{\alpha}(B_{\alpha 2})$ is a nonsingular algebraic surface in A_{α}, containing all the Y_{1r}^{α} and Y_{2s}^{α}. By construction, X_{α} is an algebraic model of $X_1 \# X_2$ and

$$\textit{order } H_1^{\text{alg}}(X_{\alpha}, \mathbb{Z}/2) \geq k_1 k_2 = 2^{\beta_1 + \beta_2}.$$

The homology classes $[h_\alpha(N_{1t}^\alpha)]_{X_\alpha}$ and $[h_\alpha(N_{2u}^\alpha)]_{X_\alpha}$ do not belong to $H_1^{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2)$. Indeed, let $M_\alpha = h_\alpha(N_{1t}^\alpha)$ and suppose that $[M_\alpha]_{X_\alpha} \in H_1^{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2)$. Then by Corollary 3.2.2 there exists a diffeomorphism $g_\alpha : X_\alpha \to X_\alpha$ arbitrarily close to the identity of X_α such that $G_\alpha = g_\alpha(M_\alpha)$ is a nonsingular algebraic curve in X_α. Let

$$\pi : A_\alpha = X_1 \times X_2 \times C_\alpha \to X_1$$

be the canonical projection onto X_1. If h_α and g_α are sufficiently close to the identity mappings on $A_{\alpha 1}$ and X_α, respectively, then G_α and N_{1r}^α are diffeotopic in A_α and $[M_\alpha]_{X_\alpha} = [G_\alpha]_{X_\alpha}$. If $i : X_\alpha \to A_\alpha$ is the inclusion mapping, then by Theorem B.1.4,

$$i_*(H_1^{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2)) \subset H_1^{\mathrm{alg}}(A_\alpha, \mathbb{Z}/2)\,.$$

It follows that the class $[G_\alpha]_{X_\alpha}$ is an element of $H_1^{\mathrm{alg}}(A_\alpha, \mathbb{Z}/2)$. Since we also have that $\pi_*(H_1^{\mathrm{alg}}(A_\alpha, \mathbb{Z}/2)) \subset H_1^{\mathrm{alg}}(X_1, \mathbb{Z}/2)$, this implies that $\pi_*([G_\alpha]_{A_\alpha}) \in H_1^{\mathrm{alg}}(X_1, \mathbb{Z}/2)$. But this is impossible, since

$$\pi_*([G_\alpha]_{A_\alpha}) = \pi_*([N_{1r}^\alpha]_{A_\alpha}) = [N_{1r}]_{X_1} \notin H_1^{\mathrm{alg}}(X_1, \mathbb{Z}/2)\,.$$

It easily follows that

$$\mathit{order}\ H_1^{\mathrm{alg}}(X_\alpha, \mathbb{Z}/2) \le k_1 k_2 + l_1 l_2 = 2^{\beta_1+\beta_2} + (2^{b_1} - 2^{\beta_1})(2^{b_2} - 2^{\beta_2}).$$

Let $p_\alpha : A_\alpha \to C_\alpha$ be the canonical projection. It follows from the construction that $p_\alpha|_{X_\alpha} : X_\alpha \to C_\alpha$ is nonconstant (and, of course, regular). Thus by Theorem 4.1.2, there exists an uncountable subset $\mathcal{B}$ of $\mathcal{A}$ such that for each $\alpha, \beta \in \mathcal{B}$, $\alpha \neq \beta$, the varieties X_α and X_β are nonisomorphic. $\square$

Corollary 4.4.3 *Let X_1 and X_2 be two compact connected nonsingular affine real algebraic surfaces. If $\beta(X_2) = b(X_2)$ (i.e. $H^1_{alg}(X_2, \mathbb{Z}/2) = H^1(X_2, \mathbb{Z}/2)$), then there exists an uncountable family of mutually nonisomorphic algebraic models $\{X_\alpha\}_{\alpha \in A}$ of the connected sum $X_1 \# X_2$ such that*

$$\beta(X_\alpha) = \beta(X_1) + b(X_2).$$

Theorem 4.4.4 *If (k, g) is a pair of integers with $0 \leq k \leq 2g$, then there exists an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of mutually nonisomorphic algebraic models of a compact connected smooth orientable surface M of genus g, with $\beta(X_\alpha) = k$.*

Proof. By Theorem 4.4.1 (i) there exists a nonsingular real algebraic orientable surface X_1 of genus g with $\beta(X_1) = k$. Applying Corollary 4.4.3 to X_1 and $X_2 = S^2$, one obtains uncountably mutually nonisomorphic surfaces X_α diffeomorphic to $X_1 \# X_2$, and thus to M, with $\beta(X_\alpha) = k$. □

Theorem 4.4.5 *Let (k, g) be a pair of integers such that $1 \leq k \leq g$. Then there exists an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraic models of a compact connected smooth nonorientable surface M of genus g, with $\beta(X_\alpha) = k$.*

Proof. Let X_1 be a nonsingular real algebraic surface diffeomorphic to a compact connected nonorientable surface of genus g with $\beta(X_1) = k$ (cf. Theorem 4.4.1 (i)). Then applying Corollary 4.4.3 to X_1 and $X_2 = S^2$, the theorem follows. □

Theorem 4.4.6 *Let M be a compact connected smooth nonorientable surface of even genus g. Let k, m be two integers with*

$$\begin{aligned} & k = m + 1 \ \textit{and} \ 2 \leq k \leq g, \\ \textit{or} \ \ & k = m \ \textit{and} \ 1 \leq k \leq g - 1. \end{aligned}$$

Then there exists an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraic models of M, such that $\beta(X_\alpha) = k$ and $\delta(X_\alpha) = m$.

Proof. By Theorem 4.4.1 (ii), one can find in each case an algebraic model X_1 of a compact nonorientable smooth surface of genus g such that $\beta(X_1) = k$ and $\delta(X_1) = m$. Now apply Corollary 4.4.3 to X_1 and $X_2 = S^2$. Then one obtains an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraic models of M, with $\beta(X_\alpha) = k$. For topological reasons, we have $\delta(X_\alpha) = m$. □

5 Two applications

5.1 Approximation by regular mappings into S^1

Let X and Y be nonsingular affine real algebraic varieties. We denote by $C^\infty_{\mathcal{R}}(X,Y)$ the closure of the set $\mathcal{R}(X,Y)$ in $C^\infty(X,Y)$. If X is compact and $Y=\mathbb{R}$, then by the Weierstrass approximation theorem, $C^\infty_{\mathcal{R}}(X,\mathbb{R}) = C^\infty(X,\mathbb{R})$, that is, $\mathcal{R}(X,\mathbb{R})$ is dense in $C^\infty(X,\mathbb{R})$. This is not true for every nonsingular affine real algebraic variety Y.

For $Y = S^1, S^2, S^4$, the structure of the set $C^\infty_{\mathcal{R}}(X,Y)$ is well understood. One can find a detailed discussion of these cases in [3, Section 13.3]. In this section we will concentrate on the case $Y = S^1$ and apply the results of Chapter 4 in order to say something more about the variety X.

Consider $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as the multiplicative group of complex numbers of norm 1. Endow $C^\infty(X,S^1)$ with the induced group structure by defining for all $f,g \in C^\infty(X,S^1)$,

$$(f \cdot g)(x) = f(x)\,g(x) \quad \text{for all} \;\; x \in X.$$

Then $\mathcal{R}(X,S^1)$ and $C^\infty_{\mathcal{R}}(X,S^1)$ are subgroups of $C^\infty(X,S^1)$ and one can define the quotient

$$\Gamma(X) = C^\infty(X,S^1)\,/\,C^\infty_{\mathcal{R}}(X,S^1)\,.$$

The group $\Gamma(X)$ measures the size of the set $C^\infty_{\mathcal{R}}(X,S^1)$ of smooth mappings which are approximable by regular mappings. In particular, $\mathcal{R}\left(X,S^1\right)$ is dense in $C^\infty\left(X,S^1\right)$ if and only if $\Gamma(X)=0$.

Let M be a compact connected smooth manifold. Denote by r the "restriction mod 2" homomorphism

$$r : H^1(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to H^1(M,\mathbb{Z}/2)$$

and let

$$A(M) = r\left(H^1(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\right).$$

The structure of the group $\Gamma(X)$ is described with the following theorem [3, Thm. 13.3.5].

Theorem 5.1.1 *Let X be a compact connected nonsingular affine real algebraic variety. Then*

$$\Gamma(X) \simeq A(X) / \left(A(X) \cap H^1_{\mathrm{alg}}(X,\mathbb{Z}/2)\right).$$

Corollary 5.1.2 *Let M be a compact connected smooth manifold of dimension greater than 1, and let*

$$\alpha(M) = \begin{cases} \operatorname{rank}\left(H^1(M,\mathbb{Z})\right) - 1 & \text{if } M \text{ is nonorientable and} \\ & w_1(M) \in A(M) \\ \operatorname{rank}\left(H^1(M,\mathbb{Z})\right) & \text{otherwise} \end{cases}$$

Then:

(*i*) *For each algebraic model X of M, one has $\Gamma(X) \simeq (\mathbb{Z}/2)^s$ for some integer s satisfying $0 \le s \le \alpha(M)$.*

(*ii*) *For each integer s satisfying $0 \le s \le \alpha(M)$, there exists an algebraic model X of M with $\Gamma(X) \simeq (\mathbb{Z}/2)^s$.*

For a proof see [3, Cor. 13.3.6]. With the results of Chapter 4 we now can prove the following

Theorem 5.1.3 *Let M be a compact connected smooth manifold of dimension ≥ 2. Then for each integer s satisfying $0 \leq s \leq \alpha(M)$, there exists an uncountable family $\{X_\alpha\}_{\alpha \in A}$ of mutually nonisomorphic algebraic models of M such that $\Gamma(X_\alpha) \simeq (\mathbb{Z}/2)^s$.*

Proof. Let G be a subgroup of $H^1(M, \mathbb{Z}/2)$ containing $w_1(M)$. Then by Theorems 4.3.2 and 4.4.4-6, we can find an uncountable family $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of mutually nonisomorphic algebraic models of M such that for each $\alpha \in A$,

$$A(M) / (A(M) \cap G) \simeq A(X_\alpha) / \left(A(X_\alpha) \cap H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2)\right).$$

The space $A(M)$ is a $\mathbb{Z}/2$-vector space, and by Theorem 5.1.1, $\Gamma(X_\alpha)$ is isomorphic to $A(X_\alpha)/(A(X_\alpha) \cap H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2))$. Therefore, $\Gamma(X_\alpha) \simeq (\mathbb{Z}/2)^s$ for some integer s with $0 \leq s \leq \alpha(M)$. Since we have taken G arbitrarily, the theorem follows. $\square$

5.2 Noetherian factorial rings

Let X be a compact irreducible nonsingular affine real algebraic variety. By [3, Cor. 12.2.6] the ring $\mathcal{R}(X)$ of regular real valued functions on X is factorial if and only if every algebraic line bundle over X is algebraically trivial. And from [3, Thm. 12.4.6] we know that $V^1_{\text{alg}}(X)$ is isomorphic to $H^1_{\text{alg}}(X, \mathbb{Z}/2)$. Therefore, $\mathcal{R}(X)$ is factorial if and only if $H^1_{\text{alg}}(X, \mathbb{Z}/2) = 0$.

Now let M be a smooth curve and let X be an algebraic model of M. Then $H^1_{\text{alg}}(X, \mathbb{Z}/2) \neq 0$ and thus the ring $\mathcal{R}(X)$ is not factorial. If X is an algebraic model of a nonorientable compact connected smooth manifold M, then $\mathcal{R}(X)$ is not factorial either, since in that case, $w_1(X) \neq 0$.

Theorem 5.2.1 *For each $n \geq 2$ there exists an uncountable family of mutually nonisomorphic Noetherian factorial rings $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of Krull dimension n such that each A_α has the property that $A_\alpha/\mathfrak{m} \simeq \mathbb{R}$ for each maximal ideal $\mathfrak{m}$ of A_α.*

Proof. Let M be an orientable compact connected smooth manifold of dimension $n \geq 2$. By Theorem 4.3.2 and Section 4.4, there exist uncountably many mutually nonisomorphic algebraic models $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of M with $H^1_{\text{alg}}(X_\alpha, \mathbb{Z}/2) = 0$ for each $\alpha \in \mathcal{A}$. Then by [3, Cor. 12.2.6], the ring $\mathcal{R}(X_\alpha)$ is factorial. Since each $\mathcal{R}(X_\alpha)$ is Noetherian and the dimension of M is equal to the Krull dimension of $\mathcal{R}(X_\alpha)$, this proves the theorem. $\square$

Appendix A

Proofs of Theorem 2.1.6 and Corollary 2.1.7

In this appendix we give a few results concerning Thom classes, which are needed for the proofs of Theorem 2.1.6 and Corollary 2.1.7. The exposition and proofs are taken from unpublished lecture notes by Bochnak and Kucharz [10].

A.1 Thom classes

Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be topological subspaces. Let $f : (X, A) \to (Y, B)$ be continuous mapping, then f induces the homomorphisms

$$f_* : H_p(X, A; \mathbb{Z}/2) \to H_p(Y, B; \mathbb{Z}/2) \quad \text{and}$$
$$f^* : H^p(Y, B; \mathbb{Z}/2) \to H^p(X, A; \mathbb{Z}/2).$$

If $f : (X, A) \to (Y, B)$ is the inclusion mapping, then we call f^* the restriction homomorphism.

Remark A.1.1 *If $m \geq 0$, then*

$$H_p(\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}; \mathbb{Z}/2) = 0 \ \text{ if } \ p \neq m \text{ and } H_m(\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}; \mathbb{Z}/2) = \mathbb{Z}/2$$
$$H^p(\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}; \mathbb{Z}/2) = 0 \ \text{ if } \ p \neq m \text{ and } H^m(\mathbb{R}^m, \mathbb{R}^m \backslash \{0\}; \mathbb{Z}/2) = \mathbb{Z}/2$$

A proof of this well known fact can be found in [29, p. 196].

Let X be a topological space and let $\xi = (E, \pi, X)$ be a real vector bundle of rank k on X. Denote by O_E the image of the zero section $X \to E$ which assigns to each $x \in X$ the zero element 0_{E_x} of the fibre $E_x = \pi^{-1}(x)$. It follows from Remark A.1.1 that

$$H^k(E_x, E_x \backslash \{0_{E_x}\}) = \mathbb{Z}/2.$$

Theorem A.1.2 *There exists a unique element* $\tau_\xi \in H^k(E, E \backslash O_E; \mathbb{Z}/2)$ *whose restriction to* $(E_x, E_x \backslash \{0_{E_x}\})$ *generates* $H^k(E_x, E_x \backslash \{0_{E_x}\})$ *for each* $x \in X$. *Furthermore, for each integer* q, *the homomorphism*

$$\varphi_q : H^q(X; \mathbb{Z}/2) \to H^{k+q}(E, E \backslash O_E; \mathbb{Z}/2)$$

defined by

$$\varphi_q(v) = \pi^*(v) \cup \tau_\xi \quad \textit{for all} \quad v \in H^q(X; \mathbb{Z}/2),$$

is an isomorphism.

For a proof see [28, p. 106]. The element τ_ξ is called the *Thom class* of ξ and φ_q is called the *Thom isomorphism.*

Let M be a smooth manifold and let $N \subset M$ be a smooth submanifold. Let $\xi = (E, \pi, X)$ be a tubular neighborhood of N in M. By the excision property, the inclusion mapping $e : (E, E \backslash N) \to (M, M \backslash N)$ induces an isomorphism of cohomology groups

$$e^* : H^k(M, M \backslash N; \mathbb{Z}/2) \to H^k(E, E \backslash N; \mathbb{Z}/2).$$

Hence there is a unique cohomology class $\tau_N^M \in H^k(M, M \backslash N; \mathbb{Z}/2)$ such that $e^*\left(\tau_N^M\right) = \tau_\xi$. In other words, τ_N^M is a unique cohomology class in the group $H^k(M, M\backslash N; \mathbb{Z}/2)$ whose restriction to $(E_x, E_x\backslash\{x\})$ generates the cohomology group $H^k(E, E\backslash\{x\}; \mathbb{Z}/2)$ for each point $x \in N$.

By Theorem A.1.2, the homomorphism

$$\psi_q : H^q(N; \mathbb{Z}/2) \to H^{k+q}(M, M\backslash N; \mathbb{Z}/2),$$
$$\psi_q(\alpha) = (e^*)^{-1}(\pi^*(\alpha) \cup \tau_\xi) \quad \text{for all} \quad \alpha \in H^q(N; \mathbb{Z}/2)$$

is an isomorphism for each integer q.

Since any two tubular neighborhoods of N in M are isotopic [20, p. 112], the cohomology class τ_N^M and the isomorphisms ψ_q do not depend on the choice of the tubular neighborhood ξ. The cohomology class τ_N^M is called the *Thom class of N in M*, and ψ_q is called the *Thom isomorphism.*

A.2 Some technical results

We will need a few technical results before we can prove Theorem 2.1.6 and Corollary 2.1.7. We begin with the naturality property of the cap product [29, p. 388].

Proposition A.2.1 (Naturality) *Let A_1 and A_2 be open subsets of a topological space X, and let B_1 and B_2 be open subsets of a topological space Y. Assume that*

$$f : (X, A_1 \cup A_2) \to (Y, B_1 \cup B_2)$$

is a continuous mapping satisfying $f(A_i) \subset B_i$ and let

$$f_i : (X, A_i) \to (Y, B_i)$$

be the mapping defined by f $(i = 1, 2)$. *If* $u \in H_{p+q}(X, A_1 \cup A_2; \mathbb{Z}/2)$ *and* $\beta \in H^q(Y, B_2; \mathbb{Z}/2)$, *then*

$$(f_1)_* (f_2^* (\beta) \cap u) = \beta \cap f_* (u),$$

where $f_2^* (\beta) \cap u \in H_p(X, A_1; \mathbb{Z}/2)$ *and* $\beta \cap f_* (u) \in H_p(X, B_1; \mathbb{Z}/2)$.

Lemma A.2.2 *Let* M *be a smooth manifold and let* N *be a closed smooth submanifold of* M. *If* $U \subset M$ *is an open subset and if*

$$i : (U, U \backslash (U \cap N)) \to (M, M \backslash N)$$

is the inclusion mapping, then

$$i^* \left(\tau_N^M\right) = \tau_{U \cap N}^U.$$

Proof. If $N \subset U$, the conclusion follows directly from the definition of the Thom class.

Let us consider the general case. Let $\xi = (E, \pi, N)$ be a tubular neighborhood of N in M. Setting $V = \pi^{-1}(U \cap N)$, we obtain that $V \cap N = U \cap N$. Again, it follows from the definition of the Thom class that

$$j^* \left(\tau_N^M\right) = \tau_{U \cap N}^V,$$

where $j : (V, V \backslash (V \cap N)) \to (M, M \backslash N)$ is the inclusion mapping. Furthermore, by the observation at the beginning of the proof,

$$j_1^* (\tau_{U \cap N}^V) = \tau_{U \cap N}^{U \cap V}, \quad j_2^* (\tau_{U \cap N}^U) = \tau_{U \cap N}^{U \cap V},$$

where

$$j_1 : (U \cap V, (U \cap V) \backslash (U \cap N)) \to (V, V \backslash (V \cap N)) \quad \text{and}$$
$$j_2 : (U \cap V, (U \cap V) \backslash (U \cap N)) \to (U, U \backslash (U \cap N))$$

are the inclusion mappings. Since $j \circ j_1 = i \circ j_2$, we have

$$\begin{aligned} j_2^*(\tau_{U\cap N}^U) &= j_1^*\left(j^*\left(\tau_N^M\right)\right) = (j \circ j_1)^*\left(\tau_N^M\right) \\ &= (i \circ j_2)^*\left(\tau_N^M\right) = j_2^*\left(i^*\left(\tau_N^M\right)\right). \end{aligned}$$

By the excision property, j_2^* is an isomorphism, hence $i^*\left(\tau_N^M\right) = \tau_{U\cap N}^U$.

□

Lemma A.2.3 *Let M be a smooth manifold and let N be a closed smooth submanifold of M. Let $\{U_\lambda\}$ be a cover of N where each U_λ is an open subset of M. Let $i_\lambda : (U_\lambda, U_\lambda\backslash(U_\lambda \cap N)) \to (M, M\backslash N)$ be the inclusion mapping. If τ is an element of $H^k(M, M\backslash N; \mathbb{Z}/2)$ such that $i_\lambda^*(\tau) = \tau_{U_\lambda \cap N}^{U_\lambda}$ for each λ, then $\tau = \tau_N^M$.*

Proof. Let $\xi = (E, \pi, N)$ be a tubular neighborhood of N in M. Let $x \in N$ and choose λ such that $x \in U_\lambda \cap N$. Denote by

$$\begin{aligned} j &: (E_x, E_x\backslash\{x\}) \to (M.M\backslash N), \\ j_1 &: (U_\lambda \cap E_x, U_\lambda \cap (E_x\backslash\{x\})) \to (E_x, E_x\backslash\{x\}), \\ j_2 &: (U_\lambda \cap E_x, U_\lambda \cap (E_x\backslash\{x\})) \to (U_\lambda, U_\lambda\backslash(U_\lambda \cap N)) \end{aligned}$$

the inclusion mappings. By Lemma A.2.2, we have $i_\lambda^*(\tau) = i_\lambda^*\left(\tau_N^M\right)$, and hence $j_2^*(i_\lambda^*(\tau)) = j_2^*\left(i_\lambda^*\left(\tau_N^M\right)\right)$. Since $i_\lambda \circ j_2 = j \circ j_1$, we have that $j_2^* \circ i_\lambda^* = j_1^* \circ j^*$, which implies $j_1^*(j^*(\tau)) = j_1^*\left(j^*\left(\tau_N^M\right)\right)$. By the excision property, j_1^* is an isomorphism, and therefore $j^*(\tau) = j^*\left(\tau_N^M\right)$. It follows from the definition of the Thom class of a submanifold that $\tau = \tau_N^M$. □

Lemma A.2.4 *Let M be a smooth manifold and let N be a closed smooth submanifold of M. Let $\xi = (E, \pi, N)$ be a tubular neighborhood of N in M. Suppose that K is a compact subset of N and W an open subset of E such that $N \backslash K \subset W \subset E \backslash \pi^{-1}(K)$. Then*

$$\bar{\pi}_* \left(\tau_K^E \cap o_K^E \right) = o_K^N,$$

where $\bar{\pi} : (E, W) \to (N, N \backslash K)$ is the mapping defined by π, and $\tau_K^E \cap o_K^E$ is an element of the group $H_n(E, W; \mathbb{Z}/2)$.

Proof. The proof of the lemma consists of three parts.

PART 1. Assume that $K = \{x\}$ and that ξ is trivial. Then we may also assume that $E = N \times \mathbb{R}^k$, $\pi : E \to N$ is the canonical projection, and N is identified with $N \times \{0\} \subset E$. Let o and τ be the unique generators of $H_k(\mathbb{R}^k, \mathbb{R}^k \backslash \{0\}; \mathbb{Z}/2)$ and $H^k(\mathbb{R}^k, \mathbb{R}^k \backslash \{0\}; \mathbb{Z}/2)$, respectively. The Thom class of ξ is equal to $1_N \times \tau$, where $\times$ denotes the cohomology cross product and $1_N = c^N(1)$, with $c^N : \mathbb{Z}/2 \to H^0(N, \mathbb{Z}/2)$ being the augmentation homomorphism (cf. [28, p. 106]). Since $\tau_N^E = \tau_\xi$, it follows that $\tau_N^E = 1_N \times \tau$. Furthermore, $o_{(x,0)}^E = o_x^N \times o$. Then

$$\begin{aligned} \tau_N^E \cap o_{(x,0)}^E &= (1_N \times \tau) \cap \left(o_x^N \times o \right) \\ &= \left(1_N \cap o_x^N \right) \times (\tau \cap o) = o_x^N \times (\tau \cap o). \end{aligned}$$

It follows from the universal coefficient theorem [28, p. 259] that $\tau \cap o$ is the unique generator of $H_0(\mathbb{R}^k, \mathbb{Z}/2)$, hence

$$\bar{\pi}_* \left(\tau_V^U \cap o_x^U \right) = \bar{\pi}_* \left(o_x^V \times (\tau \cap o) \right) = c_{\mathbb{R}^k} \left((\tau \cap o) \right) o_x^N = o_x^U,$$

where $c_{\mathbb{R}^k} : H_0(\mathbb{R}^k, \mathbb{Z}/2) \to \mathbb{Z}/2$ is the augmentation homomorphism.

PART 2. Assume that $K = \{x\}$ and ξ is not necessarily trivial. Choose a neighborhood V of x in N such that the restriction of ξ to V is trivial. Set $U = \pi^{-1}(V)$, then by Part 1, we obtain

$$\tilde{\pi}_* \left(\tau_V^U \cap o_x^U\right) = o_x^U,$$

where $\tilde{\pi} : \left(U, U\backslash\pi^{-1}(x)\right) \to (V, V\backslash\{x\})$ is the mapping defined by π. Let

$$\begin{aligned} i &: (U, U\backslash\{x\}) \to (E, E\backslash\{x\}), \\ i_1 &: \left(U, U\backslash\pi^{-1}(x)\right) \to \left(E, E\backslash\pi^{-1}(x)\right), \text{ and} \\ i_2 &: (U, U\backslash V) \to (E, E\backslash N) \end{aligned}$$

be the inclusion mappings. Clearly,

$$U\backslash\{x\} = \left(U\backslash\pi^{-1}(x)\right) \cup (U\backslash V) \quad \text{and} \quad E\backslash\{x\} = \left(E\backslash\pi^{-1}(x)\right) \cup (E\backslash N).$$

By Lemma A.2.2, $i_2^*\left(\tau_N^E\right) = \tau_V^U$ (note that $V = U \cap N$). Hence making use of Proposition A.2.1, we obtain

$$(i_1)_* \left(\tau_V^U \cap o_x^U\right) = (i_1)_* \left(i_2^*\left(\tau_N^E\right) \cap o_x^U\right) = \tau_N^E \cap i_*\left(o_x^U\right) = \tau_N^E \cap o_x^E.$$

Note that $e \circ \tilde{\pi} = \bar{\pi} \circ i_1$, where $e : (V, V\backslash\{x\}) \to (N, N\backslash\{x\})$ is the inclusion mapping. Thus

$$\begin{aligned} \bar{\pi}_* \left(\tau_V^E \cap o_x^E\right) &= \bar{\pi}_* \left((i_1)_* \left(\tau_V^U \cap o_x^U\right)\right) \\ &= e_* \left(\bar{\pi}_* \left(\tau_V^U \cap o_x^U\right)\right) = e_*\left(o_x^V\right) = o_x^N, \end{aligned}$$

which completes the proof of Part 2.

Part 3. We shall now consider the general case. Let $x \in K$. Denote by

$$\begin{aligned} j &: (E, E \backslash N) \to (E, E \backslash \{x\}), \\ j_1 &: \left(E, E \backslash \pi^{-1}(K)\right) \to \left(E, E \backslash \pi^{-1}(x)\right), \text{ and} \\ j_2 &: (E, E \backslash N) \to (E, E \backslash N) \end{aligned}$$

the inclusion mappings. By naturality property of the cap product,

$$(j_1)_* \left(\tau_N^E \cap o_K^E\right) = (j_1)_* \left(j_2^* \left(\tau_N^E\right) \cap o_K^E\right) = \tau_N^E \cap j_* \left(o_K^E\right) = \tau_N^E \cap o_x^E.$$

Note that $j^\times \circ \bar{\pi} = \bar{\pi}^\times \circ j_1$, where $j^\times : (N, N \backslash K) \to (N, N \backslash \{x\})$ is the inclusion mapping and $\bar{\pi}^\times : \left(E, E \backslash \pi^{-1}(x)\right) \to (N, N \backslash \{x\})$ is the mapping defined by π. By Part 2, $\bar{\pi}_*^\times \left(\tau_N^E \cap o_x^E\right) = o_x^N$, and hence

$$j_*^\times \left(\bar{\pi}_* \left(\tau_N^E \cap o_x^E\right)\right) = \bar{\pi}_*^\times \left((j_1)_* \left(\tau_N^E \cap o_K^E\right)\right) = \bar{\pi}_*^\times \left(\tau_N^E \cap o_x^E\right) = o_x^N.$$

Since x is an arbitrary point of K, the lemma follows from Proposition 2.1.1. □

Proposition A.2.5 *Let M be a smooth manifold and let N be a closed smooth submanifold of M. Let K be a compact subset of N and let W be an open subset of M such that $N \backslash K \subset W \subset M \backslash K$. If $i : (N, N \backslash K) \to (M, W)$ is the inclusion mapping, then*

$$i_* \left(o_K^N\right) = \tau_N^M \cap o_K^M.$$

Proof. Let $\xi = (E, \pi, N)$ be a tubular neighborhood of N in M. There exists an open subset E' of E such that

$$N \backslash K \subset E' \subset \left(E \backslash \pi^{-1}(K)\right) \cap W,$$

and for $t \in [0,1]$ and $e \in E'$, the product te in the fibre $E_{\pi(e)}$ belongs to E'. Indeed, E' can easily be constructed by choosing a Riemannian metric on ξ and considering small discs centered at the zero element 0_{E_x} of the fibre E_x for $x \in N \backslash K$

Let $\bar{\pi} : (E, E') \to (N, N \backslash K)$ be the mapping defined by π. By Lemma A.2.4, we have

$$\bar{\pi}_* \left(\tau_N^E \cap o_N^E \right) = o_K^N,$$

where $\tau_N^E \cap o_N^E \in H_n \left(E, E'; \mathbb{Z}/2 \right)$. Define the mapping

$$\begin{aligned} &H : \left(E \times [0,1] \, , E' \times [0,1] \right) \to (M, W) \, , \\ &H \left(e, t \right) = te \quad \text{for} \quad (e,t) \in E \times [0,1] \, . \end{aligned}$$

The mapping H is a homotopy between $i \circ \bar{\pi}$ and the inclusion mapping $j_1 : (E, E') \to (M, W)$. It follows that

$$i_* \left(o_K^N \right) = i_* \left(\bar{\pi}_* \left(\tau_N^E \cap o_N^E \right) \right) = (j_1)_* \left(\tau_N^E \cap o_N^E \right).$$

Consider the inclusion mappings

$$\begin{aligned} j_2 &: (E, E \backslash N) \to (M, M \backslash N) \quad \text{and} \\ j &: (E, E \backslash K) \to (M, M \backslash K) \, . \end{aligned}$$

Note that $j_2^* \left(\tau_N^M \right) = \tau_N^E$. Hence Proposition A.2.1 implies

$$(j_1)_* \left(\tau_N^E \cap o_N^E \right) = (j_1)_* \left(j_2^* \left(\tau_N^M \right) \cap o_K^E \right) = \tau_N^M \cap j_* \left(o_K^E \right) = \tau_N^M \cap o_K^M.$$

Thus $i_* \left(o_K^N \right) = \tau_N^M \cap o_K^M$. $\square$

Theorem A.2.6 *Let M be a compact smooth manifold and let N be a closed smooth submanifold of M. If $e : M \to (M, M \backslash N)$ is the inclusion mapping, then $e^* \left(\tau_N^M\right)$ is Poincaré dual to $[N]_M$, that is*

$$D_M \left(e^* \left(\tau_N^M\right)\right) = [N]_M .$$

Proof. We have $e_* ([M]) = o_N^M$, and hence Proposition A.2.1 yields

$$\tau_N^M \cap o_N^M = \tau_N^M \cap e_* ([M]) = e_* \left(\tau_N^M\right) \cap [M] = D_M \left(e^* \left(\tau_N^M\right)\right) .$$

On the other hand, Proposition A.2.5 implies that $\tau_N^M \cap o_N^M = [N]_M$. Thus $D_M \left(e^* \left(\tau_N^M\right)\right) = [N]_M$. □

A.3 The proofs

Theorem 2.1.6 *Let $f : M \to N$ be a smooth mapping between the smooth compact manifolds M and N. Let Q be a closed smooth submanifold of N and let $P = f^{-1} (Q)$. Assume that f is transverse to Q. Then*

$$D_M \circ f^* \circ D_N^{-1} ([Q]_N) = [P]_M .$$

Proof. First we show that

$$\bar{f}^* \left(\tau_Q^N\right) = \tau_P^M ,$$

where $f : (M, M \backslash P) \to (N, N \backslash Q)$ is the mapping defined by f. Assume that

$$M = \mathbb{R}^p \times \mathbb{R}^k ,\ P = \mathbb{R}^p \times \{0\} ,\ N = \mathbb{R}^q \times \mathbb{R}^k ,\ Q = \mathbb{R}^q \times \{0\} ,$$

and that

$$f(x,y) = (\varphi(x,y), y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^p \times \mathbb{R}^k,$$

where $\varphi : \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^q$ is a smooth mapping. Define the mapping $g : M \to N$ by

$$g(x,y) = (0,y)$$

and let $\bar{g} : (M, M \backslash P) \to (N, N \backslash Q)$ be the mapping determined by g. Obviously, $\bar{f}$ and $\bar{g}$ are homotopic, and therefore $\bar{g}^*(\tau_Q^N) = \bar{f}^*(\tau_Q^N)$.

Consider the following diagram

$$\begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^k & \xrightarrow{g} & \mathbb{R}^q \times \mathbb{R}^k \\ p_1 \downarrow & & \downarrow p_2 \\ \mathbb{R}^p & \xrightarrow[c]{} & \mathbb{R}^q \end{array}$$

where p_1 and p_2 are the canonical projections and the mapping $c : \mathbb{R}^p \to \mathbb{R}^q$ is the constant mapping defined by $c(x) = 0$ for all $x \in \mathbb{R}^p$. The diagram commutes and for each $x \in \mathbb{R}^p$, the restriction $g : \{x\} \times \mathbb{R}^k \to \{0\} \times \mathbb{R}^k$ is a linear isomorphism. It therefore follows from the definition of the Thom class that $\bar{g}^*(\tau_Q^N) = \tau_N^M$. Thus the theorem is proved in the special case under consideration.

Let us now consider the general case. If $p \in P$, then it follows from the constant rank theorem that in suitable local coordinates around p and $f(p)$ the mapping looks like the mapping in the special case examined at the beginning of the proof. Thus there exist a family $\{U_\lambda\}$ of open subsets of M and a family $\{V_\lambda\}$ of open subsets of N such that

$$\begin{aligned} &\{U_\lambda\} \text{ is a cover of } P,\ f(U_\lambda) \subset V_\lambda, \text{ and} \\ &\bar{f}_\lambda^* \left(\tau_{V_\lambda \cap Q}^{V_\lambda} \right) = \tau_{U_\lambda \cap P}^{U_\lambda} \text{ for every } \lambda, \end{aligned}$$

where $\bar{f}_\lambda : (U_\lambda, U_\lambda \backslash (U_\lambda \cap P)) \to (V_\lambda, V_\lambda \backslash (V_\lambda \cap Q))$ is the mapping defined by f. By Lemma A.2.2, for the inclusion mappings

$$\begin{aligned} &i_\lambda : (U_\lambda, U_\lambda \backslash (U_\lambda \cap P)) \to (M, M \backslash P), \text{ and} \\ &j_\lambda : (V_\lambda, V_\lambda \backslash (V_\lambda \cap Q)) \to (N, N \backslash Q), \end{aligned}$$

we have

$$i_\lambda^* \left(\tau_P^M\right) = \tau_{U_\lambda \cap P}^{U_\lambda} \text{ and } j_\lambda^* \left(\tau_P^M\right) = \tau_{V_\lambda \cap Q}^{V_\lambda}.$$

Since $j_\lambda \circ \bar{f}_\lambda = \bar{f} \circ i_\lambda$, we obtain

$$i_\lambda^* \left(\bar{f}^* \left(\tau_Q^N\right)\right) = \bar{f}_\lambda^* \left(j_\lambda^* \left(\tau_Q^N\right)\right) = \bar{f}_\lambda^* \left(\tau_{V_\lambda \cap Q}^{V_\lambda}\right) = \tau_{U_\lambda \cap P}^{U_\lambda} = i_\lambda^* \left(\tau_P^M\right).$$

Hence Lemma A.2.3 yields $\bar{f}^*(\tau_Q^N) = \tau_P^M$, and the first part of the theorem is proved.

Assume now that the manifolds M and N are compact. Consider the inclusion mappings $e_M : M \to (M, M \backslash P)$ and $e_N : N \to (N, N \backslash Q)$. By Theorem A.2.6,

$$D_M \left(e_M^* \left(\tau_P^M\right)\right) = [P]_M, \quad D_N \left(e_N^* \left(\tau_Q^N\right)\right) = [Q]_N.$$

Since $e_N \circ f = \bar{f} \circ e_M$, making use of the first part of the theorem, we get

$$\begin{aligned} \left(D_M \circ f^* \circ D_N^{-1}\right)([Q]_N) &= \left(D_M \circ f^* \circ e_N^*\right)\left(\tau_Q^N\right) \\ &= \left(D_M \circ e_M^* \circ \bar{f}^*\right)\left(\tau_Q^N\right) \\ &= \left(D_M \circ e_M^*\right)\left(\tau_P^M\right) \\ &= [P]_M. \end{aligned}$$

Thus the theorem follows. $\square$

Corollary 2.1.7 *Let N_1 and N_2 be closed smooth submanifolds of a compact smooth manifold M. If N_1 intersects N_2 transversely, then*

$$D_M^{-1}([N_1]_M) \cup D_M^{-1}([N_2]_M) = D_M^{-1}([N_1 \cap N_2]_M).$$

Proof. Set $\alpha_i = D_M^{-1}([N_i]_M)$ for $i = 1, 2$. Since the cross product is a natural operation, we obtain

$$\begin{aligned} D_{M\times M}(\alpha_1 \times \alpha_2) &= D_M(\alpha_1) \times D_M(\alpha_2) \\ &= [N_1]_M \times [N_2]_M = [N_1 \times N_2]_{M\times M}. \end{aligned}$$

Note that the mapping $d : M \to M \times M$, $d(x) = (x, x)$ for $x \in M$, is transverse to $N_1 \times N_2$. Clearly, $d^{-1}(N_1 \times N_2) = N_1 \cap N_2$. Therefore, by Theorem 2.1.6,

$$\left(D_M \circ d^* \circ D_{M\times M}^{-1}\right)\left([N_1 \times N_2]_{M\times M}\right) = [N_1 \cap N_2]_M,$$

and hence

$$d^*(\alpha_1 \times \alpha_2) = D_M^{-1}([N_1 \cap N_2]_M)$$

Then the theorem follows from the fact that $\alpha_1 \cup \alpha_2 = d^*(\alpha_1 \times \alpha_2)$. $\square$

Appendix B

Homology classes represented by algebraic varieties and regular mappings

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In this appendix we show that for a regular mapping $f : X \to Y$ between affine compact real algebraic varieties, we have that $f_*(H_*^{\text{alg}}(X, \mathbb{Z}/2)) \subset H_*^{\text{alg}}(Y, \mathbb{Z}/2)$. We use this result from [10] for the proof of Theorem 2.4.3. We will need three lemma's for this result.

Lemma B.1.1 *Let X be a compact affine real algebraic variety and let Y be a Zariski closed subvariety of X with* $\operatorname{Sing}(X) \subset Y$ *and* $\dim Y < \dim X = n$. *For $x \in X$, let*

$$\alpha_x : H_n(X, \mathbb{Z}/2) \to H_n(X, X \backslash \{x\}; \mathbb{Z}/2)$$

be the canonical homomorphism. Let μ_1 and μ_2 be homology classes in $H_n(X, \mathbb{Z}/2)$. *If $\alpha_x(\mu_1) = \alpha_x(\mu_2)$ for all $x \in X \backslash Y$, then $\mu_1 = \mu_2$.*

Proof. Let U be a neighborhood of Y in X such that Y is a deformation retract of U. Consider the following homomorphisms

$$\alpha_U : H_n(X, \mathbb{Z}/2) \to H_n(X, U; \mathbb{Z}/2),$$
$$e_1 : H_n(X \backslash Y, U \backslash Y; \mathbb{Z}/2) \to H_n(X, U; \mathbb{Z}/2),$$

$$\beta_{U,x} : H_n(X \backslash Y, U \backslash Y; \mathbb{Z}/2) \to H_n(X \backslash Y, (X \backslash Y) \backslash \{x\}; \mathbb{Z}/2),$$
$$e_2 : H_n(X \backslash Y, (X \backslash Y) \backslash \{x\}; \mathbb{Z}/2) \to H_n(X, X \backslash \{x\}; \mathbb{Z}/2),$$

where $x \in X \backslash U$, e_1 and e_2 are the excision isomorphisms, α_U and $\beta_{U,x}$ are the canonical homomorphisms. Clearly, $\alpha_x = e_2 \circ \beta_{U,x} \circ e_1^{-1} \circ \alpha_U$. Since e_2 is an isomorphism and $\alpha_x(\mu_1) = \alpha_x(\mu_2)$, we obtain

$$\left(\beta_{U,x} \circ e_1^{-1} \circ \alpha_U\right)(\mu_1) = \left(\beta_{U,x} \circ e_1^{-1} \circ \alpha_U\right)(\mu_2). \tag{B.1}$$

Now $U \backslash Y = (X \backslash Y) \setminus (X \backslash U)$ and $X \backslash U$ is compact. Since $X \backslash Y$ is an n-dimensional manifold, it easily follows from (B.1) that

$$\left(e_1^{-1} \circ \alpha_U\right)(\mu_1) = \left(e_1^{-1} \circ \alpha_U\right)(\mu_2).$$

Hence

$$(\alpha_U)(\mu_1) = (\alpha_U)(\mu_1). \tag{B.2}$$

Now consider the commutative diagram

$$\begin{tikzcd}
0 = H_n(Y, \mathbb{Z}/2) \arrow[r] & H_n(X, \mathbb{Z}/2) \arrow[r, "\alpha"] \arrow[dr, "\alpha_U"'] & H_n(X, Y; \mathbb{Z}/2) \arrow[d, "\beta"] \\
& & H_n(X, U; \mathbb{Z}/2)
\end{tikzcd}$$

where the horizontal line is part of the long exact sequence of the pair (X, Y), and β is the canonical homomorphism. Since Y is a deformation retract of U, β is an isomorphism. Applying equation (B.2), we get $\alpha(\mu_1) = \alpha(\mu_2)$, and hence $\mu_1 = \mu_2$, since α is injective. $\square$

Lemma B.1.2 *Let X and Y be affine real algebraic varieties of dimension n with Y irreducible, and let $f : X \to Y$ be a regular mapping such that $f(X)$ is Zariski dense in Y. Then there exists a Zariski closed subvariety Σ of Y with* $\dim \Sigma < n$, *such that* (*i*) $\operatorname{Sing}(X) \subset f^{-1}(\Sigma)$, $\operatorname{Sing}(Y) \subset \Sigma$, *and* $f : X \backslash f^{-1}(\Sigma) \to Y \backslash \Sigma$ *is a local diffeomorphism.*

(*ii*) *The element* $d(f)$ *of* $\mathbb{Z}/2$, *defined by*

$$d(f) = \#\{f^{-1}(y)\}(\operatorname{mod} 2), \quad \textit{for} \ \ y \in Y \backslash \Sigma$$

does not depend on the choice of y.

Proof. Without loss of generality we may assume that X is irreducible. The mapping f induces an injection of fields $\mathcal{K}(Y) \to \mathcal{K}(X)$ of rational functions and the degree of the field extension $\mathcal{K}(X)/\mathcal{K}(Y)$ is finite, say equal to r. Replacing X with the graph $\{(x, f(x)) \in X \times Y \mid x \in X\}$ of f, we may assume that X is an algebraic subset of $\mathbb{R}^q \times \mathbb{R}^p$, $Y \subset \mathbb{R}^p$, and f is the restriction of the canonical projection $\mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p$ to X. Denote by $\bar{X}_{\mathbb{C}}$ (resp. $\bar{Y}_{\mathbb{C}}$) the Zariski complex closure of X (resp. Y) in $\mathbb{C}^q \times \mathbb{C}^p$ (resp. $\mathbb{C}^p$), and let $\bar{f}_{\mathbb{C}} : \bar{X}_{\mathbb{C}} \to \bar{Y}_{\mathbb{C}}$ be the restriction of the projection $\mathbb{C}^q \times \mathbb{C}^p \to \mathbb{C}^p$. The set

$$\left\{ y \in \bar{Y}_{\mathbb{C}} \mid \#\{\bar{f}_{\mathbb{C}}^{-1}(y)\} = r \right\}$$

contains a nonempty Zariski open subset of $\bar{Y}_{\mathbb{C}}$ [34, I, p. 63]. If $y \in Y$, the points of $\bar{f}_{\mathbb{C}}^{-1}(y) \cap \left(\bar{X}_{\mathbb{C}} \backslash X\right)$ are pairwise conjugate. Hence, there exists a nonempty Zariski open subset U of X, such that for each $y \in U$, one has that $\#\{f^{-1}(y)\} = r \,(\operatorname{mod} 2)$.

Let C be the set of critical values of the restriction of the mapping f to the set $X \backslash \left(\operatorname{Sing}(X) \cup f^{-1}(\operatorname{Sing}(Y))\right)$ and let B be the Zariski closure of $f\left(f^{-1}(C) \cup \operatorname{Sing}(X)\right)$. Now define $\Sigma = \Sigma' \cup (Y \backslash U)$, where

$$\Sigma' = B \cup (Y \backslash f(X)) \cup \operatorname{Sing} Y \subset Y,$$

then the lemma follows $\square$

Lemma B.1.3 *Let X, Y, f, and $d(f)$ be as in Lemma B.1.1. If X and Y are compact, then*

$$f_*([X]) = d(f)\,[Y]\,.$$

Proof. Choose Σ as in Lemma B.1.2 and let $y \in Y \backslash \Sigma$. Consider the commutative diagram

$$\begin{tikzcd}
H_n(X,\mathbb{Z}/2) \arrow[r, "f_*"] \arrow[d, "\alpha"'] & H_n(Y,\mathbb{Z}/2) \arrow[d, "\beta_y"] \\
H_n(X, X\backslash f^{-1}(y);\mathbb{Z}/2) \arrow[r, "\psi"'] & H_n(Y, Y\backslash\{y\};\mathbb{Z}/2)
\end{tikzcd}$$

where ψ is induced by f, and α and β_y are the canonical homomorphisms. Since $f : X\backslash f^{-1}(\Sigma) \to Y\backslash\Sigma$ is a local diffeomorphism and $y \in Y\backslash\Sigma$, it follows that

$$\psi(\alpha([X])) = d(f)\,\beta_y([Y])\,.$$

Hence we have

$$\beta_y(f_*([X])) = d(f)\,\beta_y([Y]) = \beta_y(d(f)\,[Y])\,.$$

Since the last equality holds for every y in $Y\backslash\Sigma$, it follows from Lemma B.1.1 that $f_*([X]) = d(f)\,[Y]$. $\square$

Theorem B.1.4 *Let X and Y be compact affine real algebraic varieties, and let $f : X \to Y$ be a regular mapping. Then*

$$f_*(H_*^{\mathrm{alg}}(X,\mathbb{Z}/2)) \subset H_*^{\mathrm{alg}}(Y,\mathbb{Z}/2)\,.$$

Proof. Let V be a Zariski closed irreducible subvariety of X of dimension k and let W be the Zariski closure of $f(V)$ in Y. Let $i : V \to X$ and $j : W \to Y$ be the inclusion mappings. If $\dim W < k$, then clearly $f_*(i_*([V])) = 0$. Assume therefore that $\dim W = k$. Since

$$f_* \circ i_* = j_* \circ (f|_V)_*,$$

Lemma B.1.3 applied to $f|_V : V \to W$ implies that

$$f_*(i_*([V])) = d(f|_V)\, j_*([V]).$$

thus $f_*([V]_X) \subset H_*^{\mathrm{alg}}(Y, \mathbb{Z}/2)$. $\square$

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Notations

$\mathrm{Alb}\,(X)$	Albanese variety of X, 20
$\alpha \cup \beta$	cup product between cohomology classes, 31
α_X	Albanese mapping, 20
$\mathbb{A}^n_{\mathbb{C}}\,(\mathbb{C})$	$\mathbb{C}^n$ endowed with the Zariski topology of complex algebraic sets, 5
$\mathbb{A}^n_{\mathbb{R}}\,(\mathbb{C})$	$\mathbb{C}^n$ endowed with the Zariski topology of algebraic sets defined over $\mathbb{R}$, 5
$\beta\,(X)$	dimension of $H^1_{\mathrm{alg}}\,(X, \mathbb{Z}/2)$ over $\mathbb{Z}/2$, 72
C^∞ topology	weak or strong topology, 25
$C^\infty\,(M, N)$	set of smooth mappings from M into N, 25
$C^\infty_W\,(M, N)$	$C^\infty\,(M, N)$ endowed with the weak topology, 25
$C^\infty_S\,(M, N)$	$C^\infty\,(M, N)$ endowed with the strong topology, 25
$C^\infty_{\mathcal{R}}\,(X, Y)$	closure of the set $\mathcal{R}\,(X, Y)$ in $C^\infty\,(X, Y)$, 81
$\gamma_{n,k}$	universal vector bundle of rank k over the Grassmannian, 17
D_M	Poincaré duality isomorphism, 25
$\mathrm{Diff}\,(X)$	set of all diffeomorphisms of X equipped with the C^∞ topology, 65
$\delta\,(X)$	dimension of the set $\{v \in H^1_{\mathrm{alg}}\,(X, \mathbb{Z}/2) \mid v \cup v = 0\}$ over $\mathbb{Z}/2$, 72
$\mathrm{Emb}^\infty\,(M, X)$	set of all smooth embeddings from M into X, 43
ε^n_X	trivial vector bundle of rank n on X, 16
0_{E_x}	zero element of the fibre $E_x = \pi^{-1}\,(x)$ of the vector bundle $\xi = (E, \pi, V)$ over V, 10

$\mathrm{Pic}\,(X)$	Picard group of X, 62
$\mathcal{P}\,(V)$	ring of polynomial functions, 2
$\mathbb{P}^n_{\mathbb{C}}\,(\mathbb{C})$	$\mathbb{C}P^n$ endowed with the Zariski topology of complex algebraic sets, 5
$\mathbb{P}^n_{\mathbb{R}}\,(\mathbb{C})$	$\mathbb{C}P^n$ endowed with the Zariski topology of algebraic sets defined over $\mathbb{R}$, 5
$\mathrm{Reg}\,(V)$	set of all nonsingular points of V, 4
$\mathcal{R}\,(U)$	ring of regular functions on U, 2
$\mathcal{R}_V$	sheaf of regular functions on V, 8
$\mathcal{R}(X,Y)$	set of all regular mappings from X into Y, 8
$\mathrm{Sing}\,(V)$	set of all singular points of V, 4
$\Gamma\,(X)$	$C^\infty\left(X,S^1\right)/C^\infty_{\mathcal{R}}\left(X,S^1\right)$, 81
$\Gamma^\infty\,(\xi)$	space of all smooth sections of the vector bundle ξ, 41
(T,ρ)	closed normal tubular neighborhood, 12
T_pM	tangent space to M at p, 11
$T_z^{\mathrm{Zar}}\,(V)$	Zariski tangent space of V at x, 3
τ_M	tangent bundle on M, 11
$\tau_M\vert_V$	restriction of the tangent bundle τ_M to V, 11
τ_N^M	Thom class of N in M, 87
$(V,\mathcal{R}_V)$	locally ringed space, 3
$V^1\,(M)$	set of isomorphism classes of smooth line bundles over M, 26
$w_1\,(\xi)$	first Stiefel-Whitney class of the vector bundle ξ, 26
$[X]$	fundamental class of the algebraic variety X, 32
$X_{\mathbb{C}}$	complex algebraic variety X defined over $\mathbb{R}$ regarded as defined over $\mathbb{C}$, 62
$X'_{\mathbb{C}}$	nonsingular projective complexification of X, 10
$\bar{X}_{\mathbb{C}}$	Zariski complex closure of X in $\mathbb{C}^p$, 101
$\xi=(E,\pi,V)$	real vector bundle over V, 10

$\xi \oplus \eta$	Whitney sum of vector bundles, 17
$\xi \otimes \eta$	tensor product of vector bundles, 26
ψ_q	Thom isomorphism, 87
$[Z]_X$	homology class of the algebraic variety X represented by an algebraic subset Z of X, 33
$\Omega^1(X)$	space of holomorphic 1-forms on X, 19

Index

Samenvatting

Niet-isomorfe algebraïsche modellen van gladde variëteiten

Een niet-singuliere affiene reële algebraïsche variëteit kan men beschouwen als een gladde variëteit. Als M een gladde variëteit is en X een niet-singuliere affiene reële algebraïsche variëteit diffeomorf met M, dan noemt men X een *algebraïsch model van M*. Volgens de stelling van Tognoli heeft iedere compacte gladde variëteit een algebraïsch model [38]. Deze stelling, die Tognoli in 1973 publiceerde, was een generalisatie van een eerder resultaat van Nash, die in 1952 had bewezen dat iedere compacte samenhangende gladde variëteit diffeomorf is met een samenhangende component van een niet-singuliere reële algebraïsche verzameling [37]. In 1991 konden Bochnak en Kucharz bewijzen dat iedere compacte gladde variëteit zelfs overaftelbaar veel onderling niet-isomorfe, algebraïsche modellen heeft [6].

De stelling van Tognoli inspireerde Bochnak en Kucharz ook tot het volgende resultaat [5]. Als M een compacte samenhangende gladde variëteit van dimensie ≥ 3 is en als G een ondergroep is van $H^1(M, \mathbb{Z}/2)$ die de eerste Stiefel-Whitney klasse $w_1(M)$ bevat, dan bestaat er een algebraïsch model X van M zo dat G isomorf is met $H^1_{\text{alg}}(X, \mathbb{Z}/2)$, de groep van cohomologie klassen gerepresenteerd door algebraïsche cycels van co-dimensie 1 in X.

In dit proefschrift bewijzen we een variant hierop door een overaftelbare familie van onderling niet-isomorfe algebraïsche modellen van M met deze eigenschap te construeren (Stelling 4.3.2). We bekijken ook het geval dat M een oppervlak is en vinden een soortgelijk resultaat in Paragraaf 4.4.

In Paragraaf 5.1 passen we de resultaten van Hoofdstuk 4 toe. We bekijken de verzameling $C^\infty_\mathcal{R}\left(X, S^1\right)$ van alle gladde afbeeldingen van een reële algebraïsche variëteit X naar de eenheidscirkel S^1 die kunnen worden benaderd door reguliere afbeeldingen. We bepalen de grootte van deze verzameling voor overaftelbaar veel niet-isomorfe algebraïsche modellen X_α van een gladde varíëteit M (Stelling 5.1.3).

Stelling 5.2.1 geeft een andere toepassing van de resultaten van Hoofdstuk 4. We bewijzen dat er voor iedere $n \geq 2$ een overaftelbare familie van onderling niet-isomorfe Noetherse ontbindingsringen $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ van Krull dimensie n bestaat, zo dat iedere A_α de eigenschap heeft dat $A_\alpha/\mathfrak{m} \simeq \mathbb{R}$ voor ieder maximaal ideaal $\mathfrak{m}$ van A_α.

De theorie die nodig is voor de bovenstaande resultaten wordt behandeld in de eerste drie hoofdstukken van dit proefschrift. Hoofdstuk 4 bevat de hoofdresultaten en in Hoofdstuk 5 worden de twee toepassingen gegeven. In Appendix A geven we enkele resultaten met betrekking tot Thom klassen welke nodig zijn voor de bewijzen van Stelling 2.1.6 en Gevolg 2.1.7 in Paragraaf A.3. Een resultaat dat we gebruiken in het bewijs van Stelling 2.4.3 wordt in Appendix B bewezen.